Wizards vs. Time Machines

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Outline

1. What is Complexity Theory?
    - Models of computation
    - Complexity classes
    - Interactive Proofs
    - Closed Timelike Curves

Decision problems

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For example, let $\Sigma = \{(,), 0, 1\}$ and let MATRIX MULTIPLICATION be the set of strings (A, B, C) for which each of A, B, C is an n^2-length list of binary strings, and $AB = C$ when these are interpreted as $n \times n$ matrices of binary integers.

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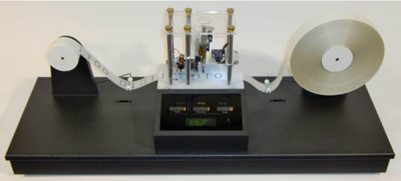
- Informally, we say that a process *decides* L if it has some behavior for $x \in L$ and a different behavior for $x \notin L$.
- A *model of computation* is a way to specify what kind of thing the computational process can be.

A precise model of computation

Definition (Turing Machine)

A *Turing machine* T consists of...



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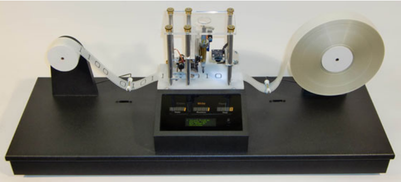

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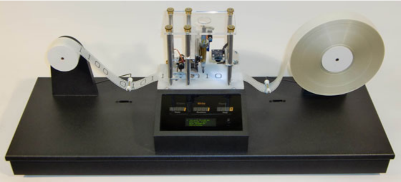

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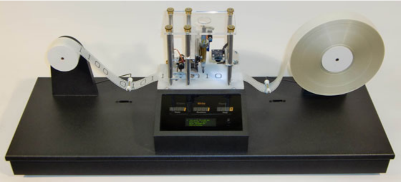

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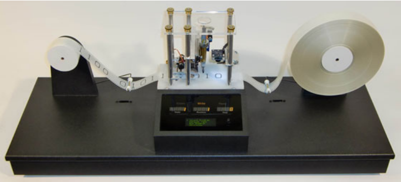

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Computation *halts* when the machine enters the *accept state* or the *reject state*.

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P We say that $L \in \mathrm{P}$ or L is *decidable in polynomial time* if there is some polynomial p and Turing machine T such that T decides L in time p.

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In particular, any two sufficiently powerful models of a computer can simulate each other in polynomial time.
P would be the same if we replace our Turing machine with a multi-tape Turing machine, a Python program, DNA-based computation, etc.

A problem in P

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The standard matrix multiplication algorithm for two $n \times n$ matrices takes about n^3 arithmetic operations. Implement this algorithm in your favorite programming language. $\square$

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Randomness as a resource

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- If $x \in L$, then $M(x, r)$ accepts with probability at least $\frac{2}{3}$.

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- $x^4 + y^4 + (x + y)^4$ is identically 0,
- while $x^3 + y^3 + (x + y)^3 = x^2y + xy^2$ is not identically 0

A randomized algorithm

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We'll refer to w variously as a witness, proof, or certificate.

Example

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If $G \cong H$, then the situation is identical from Merlin's point of view, regardless of which graph you picked. He will be right with probability exactly $\frac{1}{2}$

AM

Definition

We say a language is in AM if there is a randomized poly-time algorithm A, a function M, and a verifier V such that when x is an input of length at most n,

- If $x \in L$, then $V(x, A(x), M(x, A(x)))$ accepts with probability at least $1 - \varepsilon$

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We say that a Turing machine T decides L in space s if T decides L and whenever T is run on an input of length at most n, it touches only $s(n)$ squares on its tape.

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Idea: Polynomial space is big enough to do *brute-force search*.

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Suppose that $L \in$ NP. Let V be a verifier and let q be a polynomial such that for $x \in L$ of length at most n, there is a witness w of length at most $q(n)$.

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5. Otherwise, increment w and return to step 2.

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Proof.

Given a PSPACE-algorithm for problem L, switch the accept and reject states. This is a PSPACE-algorithm for $\overline{L}$. $\square$

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In our previous brute force search, we only cared about finding a single point in the search space with a specified property. PSPACE-computations can do much more than that, however.

Proof.

Proof that BPP $\subseteq$ PSPACE Let $L \in$ BPP with algorithm M such that $\Pr_r[M(x, r) accepts] \geq \frac{2}{3}$ for $x \in L$ and $\Pr_r[M(x, r) accepts] \leq \frac{1}{3}$ for $x \notin L$. We give a PSPACE-algorithm deciding M:

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4. If the accept counter is larger, halt and accept. Otherwise

Outline

1. What is Complexity Theory?
   - Models of computation
   - Complexity classes
   - Interactive Proofs
   - Closed Timelike Curves

CTCs

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How can we use these to do computation?

The grandfather paradox

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- Travel along the CTC until you come out 50 years before you enter.

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- Travel along the CTC until you come out 50 years before you enter.
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- Contradiction!

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This contradiction is easily resolved: set the state of the world as $\frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$!

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Theorem

If A is a Markov chain, then A has a unique $+1$-eigenvalue eigenvector, called its stable distribution. Furthermore, it has the following explicit form:

$$v = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} A^i v_0, \tag{1}$$

for any starting distribution v_0.

A $\mathrm{BPP}_{\mathrm{CTC}}$ algorithm for NP

Definition

A language L is in $\mathrm{BPP}_{\mathrm{CTC}}$ if it can be decided in polynomial time by a randomized algorithm which can find stable distributions of implicitly-defined Markov chains as a unit time operation.

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Let $A|w\rangle = |w+1\rangle$ if $V(x,w)$ rejects and $A|w\rangle = |\text{accept}\rangle$ if $V(x,w)$ accepts. Let $A|\text{accept}\rangle = |\text{accept}\rangle$.

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