Ordinal Arithmetic

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0 What's an Ordinal?

From Susan's colloquium, you know that ordinals are things that look like this:

$$
\begin{array}{rr}
0, 1, 2, 3, \ldots & \omega \\
\omega, \omega+1, \omega+2, \omega+3, \ldots & \omega 2 \\
\omega 2, \omega 3, \omega 4, \ldots & \omega^2 \\
\omega^2, \omega^3, \omega^4, \ldots & \omega^\omega \\
\omega^\omega, \omega^{\omega^\omega}, \omega^{\omega^{\omega^\omega}}, \ldots & \varepsilon_0
\end{array}
$$

In this class, we'll rigorously define objects that look like these and make sense of why we all this *shvmoop*ing gives rise to addition, multiplication, and exponentiation. We'll find, however, that these operations are order-theoretically nice but not very algebraic, so we'll explore some others.

We'll use polynomial decompositions of ordinals to define $\oplus$ and $\otimes$, which turn the space of ordinals into a commutative semiring. We'll lament that these operations don't respect limit operations, and then we'll prove a result from this year that there is no "proper" exponentation of ordinals that extends the $\oplus, \otimes$ algebraic structure.

In an attempt to salvage the limitwise failures of the semiring, we'll look at weirder operations — a new multiplication $\times$ and new exponentations $\alpha^{\times\beta}, \alpha^{\otimes\beta}$ to discover that they satisfy much nicer algebraic relations than they really ought to.

Every object we consider will be an ordinal unless otherwise stated.

Definition 0.1. An *ordinal* is a transitive set of ordinals.

Why is this not circular — don't we need ordinals to exist in order to define any sets of ordinals? (No. The empty set is a set of ordinals.)

Definition 0.2. A set X is *transitive* if $y \in X \Rightarrow y \subseteq X$.

Remark 0.3. Transitivity might seem like a funny name for this property, but it's equivalent to the property "if $a \in b \in X$, then $a \in X$".

This simple recursive definition will give rise to the entire theory, though we need one extra little detail:

Axiom 0.4 (Regularity[1])**.** *If X is any set, $X \notin X$.*

Before we delve into the theory, we should convince ourselves that some ordinals exist. Consider the set of all ordinals rigorously defined thus far — the empty set. Since the empty set has no elements, every element of the empty set is a subset of the empty set. Therefore, it is a transitive set of ordinals! Now that we have an ordinal, we can consider a new set of ordinals: $\{\emptyset\}$. Is it transitive? Yes — its only element is $\emptyset$, and $\emptyset$ is a subset of it.[2]

Now we have two new sets of ordinals to consider: $\{\{\emptyset\}\}$ and $\{\emptyset, \{\emptyset\}\}$. Are these ordinals? For the first, we have $\{\emptyset\} \in \{\{\emptyset\}\}$ but $\{\emptyset\} \not\subseteq \{\{\emptyset\}\}$ since $\emptyset \notin \{\{\emptyset\}\}$. For the second, we have $\{\emptyset\} \subseteq \{\emptyset, \{\emptyset\}\}$ and $\emptyset \subseteq \{\emptyset, \{\emptyset\}\}$, so it is a transitive set of ordinals. Since we've already considered all sets containing only the smaller ordinals, any new set of ordinals must contain the ordinal we just created. By transitivity, if it contains this ordinal, it must also contain all smaller ones. Continuing this argument, we'll calculate the first few ordinals to be

$$\emptyset, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\}, \ldots$$

This suggests the following definition:

Definition 0.5. If α is an ordinal, then its *successor* is $\alpha + 1 = \alpha \cup \{\alpha\}$. We will say β is a *successor ordinal* if there is some α for which $\beta = \alpha + 1$.

Proposition 0.6. *If α is an ordinal, then $\alpha + 1$ is an ordinal.*

Proof. We know $\alpha \cup \{\alpha\}$ is a set of ordinals, so it just remains to check that $\alpha \cup \{\alpha\}$ is transitive. Assume $\beta \in \alpha \cup \{\alpha\}$. If $\beta \in \alpha$, then $\beta \subseteq \alpha \subseteq \alpha \cup \{\alpha\}$. Otherwise, $\beta = \alpha \subseteq \alpha \cup \{\alpha\}$. $\square$

Yay! Our notation here is suggestive. Let's use it to rename ordinals reachable by finite applications of the successor operation: (these turn out to be all of the finite ordinals — can you prove it?)

$$0 = \emptyset, 1 = 0 + 1 = \{0\}, 2 = 1 + 1 = \{0, 1\}, 3 = 2 + 1 = \{0, 1, 2\}, \ldots$$

Notice that the number we use to label an ordinal is the same as the number of elements it contains. By induction, the ordinals reachable by finite applications of the successor function are precisely the natural numbers, where we identify an ordinal with the natural number we've just used to label it.

Consider again the set of all ordinals explicitly defined thus far, this time: $\omega = \{0, 1, 2, \ldots\}$.

[1] When most logicians say "the axiom of regularity", they're referring to a more powerful statement of which this is a consequence. We don't need to worry about it.

[2] Note that we don't have to check anything here — $\emptyset \subseteq X$ is true for any set X.

Proposition 0.7. *ω is an ordinal which is not a successor.*

Proof. We'll use "natural number" to mean "ordinal which can be constructed by finitely many applications of the successor function starting from 0". Then ω is the set of all natural numbers. If $n \in \omega$, then n is a natural number, which is a set of natural numbers, which is a subset of ω. Therefore, ω is an ordinal.

Now assume that $\omega = \beta + 1$. Then $\beta \in \omega$, so β is a natural number. Then $\beta + 1 = \omega$ is a natural number, which implies $\omega \in \omega$. Contradiction. $\square$

This argument generalizes, but we'll need more machinery before we say exactly how.

Proposition 0.8. *For all α, β, $\alpha \subseteq \beta \Leftrightarrow \alpha \in \beta$ or $\alpha = \beta$.*

Proposition 0.9. *For distinct ordinals α, β, either $\alpha \in \beta$ or $\beta \in \alpha$.*

Proof. Consider $\gamma = \alpha \cap \beta$. Now consider $\gamma + 1 = \gamma \cup \{\gamma\}$. If $\gamma + 1 \subseteq \alpha$ and $\gamma + 1 \subseteq \beta$, then $\gamma + 1 \subseteq \gamma$, which implies $\gamma \in \gamma$; contradiciton. Without loss of generality, assume $\gamma + 1 \not\subseteq \alpha$. Then we have $\gamma \subseteq \alpha$, so $(\gamma + 1) \setminus \gamma \not\subseteq \alpha$, which implies $\gamma \notin \alpha$. We know $\gamma \subseteq \alpha$, so from Proposition 0.8 we have $\gamma = \alpha$. We have $\gamma \subseteq \beta$ by the definition of intersection and $\gamma = \alpha \neq \beta$ by assumption. Applying Proposition 0.8 again, we conclude $\alpha = \gamma \in \beta$. $\square$

Definition 0.10 (Toset)**.** Let P be a set and $<$ a relation on that set. We call $<$ a *partial order* and P a *partially ordered set* or *poset* if the following are satisfied:

- For all $p \in P$, $p \not< p$
- For all $p, q, r \in P$, $p < r$ and $r < q$ implies $p < r$

Additionally, we might have the following relation:

- For all $p, q \in P$, $p < q$ or $p = q$ or $q < p$.

If that is the case, we call $<$ a *total order* and P a *totally-ordered* set or sometimes a *toset*.

Notice that any set of ordinals together with the $\in$ relation is a toset. We'll now use $\in$ interchangeably with $<$ and $\subseteq$ interchangeably with $\leq$. We'll also use words like

Definition 0.11. β is a *limit* ordinal if it is not a successor.

Now we've separated ordinals into two classes: limits and successors. The two classes turn out to be somewhat different, which is useful. If we can prove something for all successor ordinals and also for all limit ordinals, then we'll have proved it for all ordinals!

Proposition 0.12. *If X is a set of ordinals, then the union of all elements of X (which we'll denote $\bigcup_{\alpha \in X} \alpha = \bigcup X$) is an ordinal.*

Proof. If $\alpha \in \beta \in \bigcup X$, then by the definition of union there is $\gamma \in X$ such that $\alpha \in \beta \in \gamma \in X$. By transitivity of γ, we have $\alpha \in \gamma \in X$. By the definition of union, $\alpha \in \bigcup X$. Therefore, $\bigcup X$ is a transitive set of ordinals. □

Proposition 0.13. *If X is a set of ordinals with no maximum element, then $\bigcup X$ is a limit ordinal.*

Why is this the appropriate generalization of our construction of ω? We took ω just as the set of all natural numbers, but we could have taken it as the union of any increasing sequence of finite orinals, eg.

$$\bigcup_n 2^n = \bigcup \{1, 2, 4, 8, \ldots\}$$

Is 6 in this union? Recall that $8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$, so $6 \in 8$ and $6 \in \bigcup_n 2^n$. Since any natural number is bounded above by some power of 2, every natural number appears in this union. Nothing other than natural numbers appear in this union, so it is indeed ω

Proof of Proposition 0.13. Exercise. □

Proposition 0.14. *If X is a collection of ordinals, $\bigcup X$ is the least upper bound of X.*

Proof. Exercise.

□

In the topological sense of the words "sequence" and "limit", every increasing sequence of ordinals has a limit![3]

Proposition 0.15. *If X is a collection of ordinals, $\gamma = \cap X$ is the smallest ordinal in X.*

Proof. First, we need that $\gamma \in X$. By definition of intersection, $\gamma \subseteq \delta$ for all $\delta \in X$. Now consider $\gamma + 1 = \gamma \cup \{\gamma\}$. For each $\delta \in X$, either $\delta \in \gamma + 1$ or $\gamma + 1 \subseteq \delta$. If the latter were true for all δ, then we would have $\gamma \in \cap X = \gamma$, contradiction. Therefore we have $\alpha \in X$ such that $\gamma \subseteq \alpha \in \gamma + 1$ — it must be that $\alpha = \gamma$.

Finally, notice that γ must be the smallest ordinal in X, for it is a subset of each ordinal in X. □

The following is immediate, and should be familiar from Susan's colloquium:

Corollary 0.16 (Well-foundedness)**.** *There are no infinite descending chains $\alpha_1 \ni \alpha_2 \ni \alpha_3 \ni \cdots$ In other words, $\in$ is a* well-founded *relation over the ordinals.*

[3] You might ask about the sequence of all ordinals, which surely doesn't have a limit. However, this sequence doesn't exist – the class of all ordinals is not a set and therefore not a sequence.

Proposition 0.17. *β is a limit ordinal iff $\beta = \bigcup_{\alpha\in\beta} \alpha$.*

Proof. Exercise. □

From now on, we shall use intersection interchangeably with infimum and union interchangeably with supremum.

Definition 0.18. A *totally-ordered* set is a set P with a binary relation $<$ that satisfies the following properties:

- Irreflexivity: For all $p \in P$, $p \not< p$
- Transitivity: For all $p, q, s \in P$, $p < q \wedge q < s \Rightarrow p < s$
- Totality: For all $p, q \in P$, exactly one of the following is true: $p < q, p = q, q < p$.

A *well-ordered* set is a totally-ordered set in which every subset has a least element.

Theorem 0.19. *Ordinals are well-ordered sets.*

Proof. Combine Axiom 0.4, Definition 0.3, Proposition 0.9, and Proposition 0.15. □

Definition 0.20. Let $(P, <_P)$ and $(Q, <_Q)$ be ordered sets. We call $f : P \to Q$ an *order-isomorphism* if it is bijective and order-preserving, ie. $f(r) <_Q f(s)$ iff $r <_P s$.

Fact 0.21. *Any well-ordered set is order-isomorphic to a unique ordinal, called its* order-type.[4] *In other words, well-ordered sets are ordinals.*

1 Transfinite Induction

Over the natural numbers, we have this nice tool that lets us prove things:

Axiom 1.1 (Induction)**.** *Let P be a property of natural numbers. If $P(0)$ is true and $P(n) \Rightarrow P(n+1)$ is true, then $P(n)$ is true for all $n \in \mathbb{N}$.*

Equivalently, we have the following.

Axiom 1.2 (Induction, strong form)**.** *Let P be a property of natural numbers. If $\forall k < n[P(k)] \Rightarrow P(n)$, then $P(n)$ is true for all $n \in \mathbb{N}$.*

Induction is easy to believe for the following reason: for any n, we know there is a finite chain of implications we could write to prove $P(n)$. This property isn't special to the natural numbers – it applies to any well-ordered set.

Theorem 1.3 (Transfinite induction, pure form)**.** *Let P be a property of ordinals. If $\forall \alpha < \beta[P(\alpha)] \Rightarrow P(\beta)$, then $P(\beta)$ is true for all ordinal β.*

[4] In case you care about this kind of thing, this fact requires the Axiom of Choice.

Proof. Consider the collection of ordinals for which P is not true. Assuming this collection is nonempty, it has a smallest element, call it β. By definition, $P(\alpha)$ is true for all $\alpha < \beta$. Therefore, by the hypothesis of the theorem, $P(\beta)$ is true. Contradiction. □

Proving things directly with this definition is often awkward, since we have two different kinds of ordinals floating around — limits and successors. Thus the following form is often more convenient:

Corollary 1.4 (Transfinite induction, practical form)**.** *Let P be a property of ordinals. If*

- *(Base case) $P(0)$ is true,*
- *(Successor case)$P(\alpha) \Rightarrow P(\alpha+1)$ is true, and*
- *(Limit case) $\forall \alpha < \beta[P(\alpha)] \Rightarrow P(\beta)$ is true for all limit ordinals β,*

then $P(\alpha)$ is true for all ordinal α.

Now we can define things over the whole class of ordinals. We know how to get natural number addition by repeated applications of the successor function, so let's extend this to all ordinals.

Definition 1.5. *Ordinal addition.*

- For any α, $\alpha + 0 = \alpha$.
- For any α, β, $\alpha + (\beta + 1) = (\alpha + \beta) + 1$.
- For $\beta = \bigcup_{\gamma \in \beta} \gamma$ a limit ordinal, $\alpha + \beta = \bigcup_{\gamma \in \beta}(\alpha + \gamma)$.

This operation has a nice order-theoretic interpretation. If we interpret α and β as well-ordered sets, then $\alpha + \beta$ is the disjoint union $\alpha \sqcup \beta = 0 \times \alpha \cup 1 \times \beta$ with the lexicographical ("dictionary") order $(r, s) < (x, y)$ iff $r < s$ or $r = s$ and $s < y$. That is, we paste the two sets together so that all the elements of α are less than all the elements of β. This gives us intuition for the following:

Proposition 1.6. *Ordinal addition is associative. That is, $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$ for all α, β, γ.*

Proof. We'll induct on γ.

Base case $\gamma = 0$

$$(\alpha + \beta) + 0 = \alpha + \beta = \alpha + (\beta + 0).$$

Successor case Assume inductively that addition is associative when the rightmost argument is $\leq \gamma$.

$$\begin{aligned}
\alpha + (\beta + (\gamma + 1)) &= \alpha + ((\beta + \gamma) + 1) \\
\alpha + (\beta + (\gamma + 1)) &= (\alpha + (\beta + \gamma)) + 1 \\
\alpha + (\beta + (\gamma + 1)) &= ((\alpha + \beta) + \gamma) + 1 \\
\alpha + (\beta + (\gamma + 1)) &= (\alpha + \beta) + (\gamma + 1)
\end{aligned}$$

Limit case $\gamma = \bigcup_{\delta \in \gamma} \delta$

$$\begin{aligned}
(\alpha + \beta) + \gamma &= \bigcup_{\delta \in \gamma} (\alpha + \beta) + \delta \\
(\alpha + \beta) + \gamma &= \bigcup_{\delta \in \gamma} \alpha + (\beta + \delta) \\
(\alpha + \beta) + \gamma &= \bigcup_{\gamma' \in (\beta + \gamma)} \alpha + \gamma' \\
(\alpha + \beta) + \gamma &= \alpha + (\beta + \gamma)
\end{aligned}$$

□

2 Exercises

I highly recommend you think about most of these. Come to me for hints!

Exercise 2.1. Recall our definition of ordinal. Prove that the definition "an ordinal is a transitive set of sets" is not equivalent by exhibiting a transitive set of sets that is not an ordinal.

Exercise 2.2. Take the following as an alternative definition of ordinal: an ordinal is a transitive set of ordinals well-ordered by $\in$. Under this definition, prove that for all ordinals α, $\alpha \notin \alpha$.

Exercise 2.3. Prove that $\omega + 2 \neq 2 + \omega$.

Exercise 2.4. Prove that if X is a collection of ordinals with no maximum element, then $\bigcup X$ is a limit ordinal. As a corollary, prove that α is a limit ordinal iff $\alpha = \bigcup_{\gamma < \alpha} \gamma$.

Exercise 2.5. If $\beta < \beta'$, then $\alpha + \beta < \alpha + \beta'$. (Hint: Transfinite induction! You may find problem 3.1 useful.)

The next exercise is the key lemma we missed in the limit case of ordinal associativity.

Exercise 2.6. If α is a limit ordinal, then so is $\beta + \alpha$. (Hint: use the previous two exercises.)

3 Problems

If you have a good grasp of how to do the exercises, here are some fun problems. Tomorrow in class, we'll work with the following definition:

Definition 3.1. A ordinal function f is *continuous* if for every limit ordinal β, we have

$$f(\beta) = \sup_{\alpha \in \beta} f(\alpha).$$

Problem 3.2. (If the word "topology" doesn't mean anything to you, you can ignore this exercise.) Prove that an ordinal function is continuous iff it is the continuous in the topological sense with respect to the order topology. (The order topology has as basis all sets of the form $\{\gamma \mid \alpha < \gamma < \beta\}$.)

Remark 3.3. In the topological sense of the word, every increasing sequence of ordinals has a limit!

Problem 3.4. Assume f is continuous and weakly increasing, ie. $\alpha \leq f(\alpha)$ for all α. Prove that f has arbitrarily large fixed points.

Problem 3.5. The function f_α, "left addition by α", defined by $f_\alpha(\beta) = \alpha + \beta$ is continuous (we'll prove this in class tomorrow) and weakly increasing. (exercise 2.5) What can you say about its fixed points?

Definition 3.6. Let $(P, <_P)$ and $(Q, <_Q)$ be ordered sets. We call $f : P \to Q$ an *order-isomorphism* if it is bijective and order-preserving, ie. $f(r) <_Q f(s)$ iff $r <_P s$.

Problem 3.7. Prove Fact 0.21: if P is any well-ordered set, there is a unique ordinal α and map $f : P \to \alpha$ such that f is an order-isomorphism. The following are good lead-in exercises.

- Prove that for any α, the identity function is the only order-isomorphism $\alpha \to \alpha$.
- Prove that if $\alpha \neq \beta$, there is no order-isomorphism between α and β.

Hint: You can define a bijection by transfinite recursion. Say something like "if x is the least element for which f is not yet defined, define $f(x)$ to be..."

Ordinal Arithmetic Day 2

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Mathcamp 2015 W3 Friday

2 Arithmetic via Order

Recall the definition of ordinal addition.

Definition 2.1 (Ordinal addition, transfinite induction)**.**

- For any α, $\alpha + 0 = \alpha$.
- For any α, β, $\alpha + (\beta + 1) = (\alpha + \beta) + 1$.
- For $\beta = \bigcup_{\gamma \in \beta} \gamma$ a limit ordinal, $\alpha + \beta = \bigcup_{\gamma \in \beta} (\alpha + \gamma)$.

This definition is fine, and it's easy to work with since it's phrased in terms of transfinite induction. But also recall from yesterday that the ordinals are completely characterized by being the well-ordered sets. Let's relate these two facts by giving an order-theoretic interpretation of addition. First, a useful definition.

Definition 2.2. Let $(P, <_P)$ and $(Q, <_Q)$ be ordered sets. We call $f : P \to Q$ an *order-isomorphism* if it is bijective and order-preserving, ie. $f(r) <_Q f(s)$ iff $r <_P s$.

On the homework, we showed that two ordinals are equal iff they are order-isomorphic, ie. there is an isomorphism between them.

Definition 2.3 (Addition of well-ordered sets)**.** If A and B are well-ordered sets, then $A + B$ is the disjoint union

$$A \sqcup B = \{0\} \times A \cup \{1\} \times B = \{(0, a) \mid a \in A\} \cup \{(1, b) \mid b \in B\}$$

with the lexicographical order $\prec$ defined by

$$(n, x) \prec (n', x') \Leftrightarrow n < n' \text{ or } (n = n' \text{ and } x < x')$$

This is called the lexicographical or "dictionary" order because it's the same order we use to sort words: first we compare the first letter in the word, then if they're the same we compare the next letter and so on.

Proposition 2.4. *For all α, β, the ordinal sum $\alpha + \beta$ is equal to the woset sum, which for convenience we'll call $\alpha \sqcup \beta$.*

Proof. By transfinite induction on β, we shall build an order-isomorphism $f : \alpha + \beta \to \alpha \sqcup \beta$.

Base case ($\beta = 0$) $\alpha + 0 = \alpha$ and $\alpha \sqcup 0 = \{0\} \times \alpha$. Taking the Cartesian product really changes nothing, so our map will just be $f(\alpha) = (0, \alpha)$. You can check that this is an order isomorphism.

Successor case Inductively assume that we have an isomorphism $f_\beta : \alpha+\beta \to \alpha\sqcup\beta$. We shall build an isomorphism $f_{\beta+1} : \alpha+(\beta+1) \to \alpha\sqcup(\beta+1)$. Let's start by using what we already have! We'll say that for all $\gamma \in \alpha+\beta$, $f_{\beta+1}(\gamma) = f_\beta(\gamma)$. Recall that $\alpha + (\beta + 1) = (\alpha + \beta) + 1 = (\alpha + \beta) \cup \{\alpha + \beta\}$, so all we have left to define is $f(\alpha + \beta)$. Recall that $\beta + 1 = \beta \cup \{\beta\}$, so the only thing left to hit in the range is $(1, \beta)$. So let's define $f_{\beta+1}(\alpha + \beta) = (1, \beta)$, since our map needs to be bijective.

Let's check that this works (is an order-isomorphism). In other words, let's check that $f_{\beta+1}(\gamma) \prec f_{\beta+1}(\gamma')$ holds iff $\gamma < \gamma'$. By the inductive hypothesis, this is true if both are $< \beta$. All that's left to check is the case where one of them is $= \beta$.

We know that $\alpha + \beta$ is the maximal element of $\alpha + \beta + 1$ (to see this, think of it as $(\alpha + \beta) + 1$.) In formal terms, $\forall x \in (\alpha + \beta + 1)[x \leq \alpha + \beta]$. Therefore, if $(n, \gamma) \neq (1, \beta)$, then $f(n, \gamma) < f(1, \beta)$. We're done!

Limit case Inductively assume that we have an isomorphism $f_\gamma : \alpha\sqcup\beta \to \alpha+\gamma$ for all $\gamma < \beta$. Like before, β wants to be an extension of the f_γ. In fact, let's inductively assume that if $\gamma < \gamma'$, then $f_{\gamma'}$ extends f_γ. Then our work is cut out for us: let's just define f_β as the thing that extends all of the f_γ! More precisely, whenever $\delta < \beta$, there is some γ such that $\delta < \gamma < \beta$. Define $f_\beta(\delta) = f_\gamma(\delta)$. This doesn't depend on your choice of γ — since they are all extensions of each other, they take the same value on δ.

A general theme of this class is that the limit case of a transfinite induction proof involves taking a limit. But this doesn't look like a union; what gives? Formally, a function is a set of pairs (x, y) such that $f(x) = y$. With this definition, we can write $f_\beta = \bigcup_{\gamma\in\beta} f_\gamma$. □

Now that we know what addition really is, we can define multiplication and exponentiation very similarly.

Definition 2.5. *Ordinal multiplication.*

- For any α, $\alpha 0 = \alpha$.
- For any α, β, $\alpha(\beta + 1) = \alpha\beta + \alpha$.
- For β a limit ordinal, $\alpha\beta = \bigcup_{\gamma\in\beta} \alpha\gamma$.

This has a similar order theoretic interpretation: $\alpha\beta$ is the product set $\{(\gamma, \delta) \mid \gamma \in \alpha, \delta \in \beta\}$ with the lexicographical order.

Definition 2.6. *Ordinal exponentiation.*

- For any α, $\alpha^0 = 1$.

- For any α, β, $\alpha^{\beta+1} = \alpha^\beta \alpha$.
- For β a limit ordinal, $\alpha^\beta = \bigcup_{\gamma \in \beta} \alpha^\gamma$.

In order to make future proofs by transfinite induction easier, let's introduce a new definition.

Definition 2.7. A ordinal function f is *continuous* if for every limit ordinal β, we have

$$f(\beta) = \sup_{\alpha \in \beta} f(\alpha).$$

It's immediate from our definitions that addition, multiplication, and exponentiation are continuous in the right argument. That is, if we define $f_\alpha(\beta) = \alpha + \beta$, $g_\alpha(\beta) = \alpha \cdot \beta$, $h_\alpha(\beta) = \alpha^\beta$ then $f_\alpha, g_\alpha, h_\alpha$ are continuous functions. (Check this.) You can think of f_α, g_α as "left (addition/multiplication) by α" and h_α as "exponentiation in base α".

Proposition 2.8. *Ordinal exponentiation turns ordinal addition into ordinal multiplication:* $\alpha^{\beta+\gamma} = \alpha^\beta \alpha^\gamma$.

Proof. Transfinite induction on γ. Assume that ordinal multiplication is associative, which you'll prove in the homework.

Base case $\gamma = 0$

$$\alpha^{\beta+0} = \alpha^\beta = \alpha^\beta 1 = \alpha^\beta \alpha^0.$$

Successor case Assume inductively that this holds when the rightmost argument is γ.

$$\alpha^{\beta+\gamma+1} = \alpha^{\beta+\gamma}\alpha = \alpha^\beta \alpha^\gamma \alpha = \alpha^\beta \alpha^{\gamma+1}.$$

Limit case $\gamma = \bigcup_{\delta \in \gamma} \delta$ Assume inductively that this holds when the rightmost argument is $< \gamma$.

$$\begin{aligned}
\alpha^{\beta+\gamma} &= \bigcup_{\delta \in \gamma} \alpha^{\beta+\delta} \\
&= \bigcup_{\delta \in \gamma} \alpha^\beta \alpha^\delta \\
&= \bigcup_{\delta' \in \alpha^\gamma} \alpha^\beta \delta' \\
&= \bigcup_{\delta' \in \alpha^\gamma} g_{\alpha^\beta}(\delta') \\
&= g_{\alpha^\beta}\left(\bigcup_{\delta' \in \alpha^\gamma} \delta' \right) \\
&= \alpha^\beta \alpha^\gamma
\end{aligned}$$

□

3 An Application of Transfinite Induction to Things that Aren't Ordinals

Axiom 3.1 (The Well-Ordering Principle)**.** *Every set can be well-ordered. That is, if X is a set, there exists a relation $<$ on that set such that $(X, <)$ is a woset. Alternatively, there is a bijection between X and some ordinal.*

How does this work? Take a set, say $\mathbb{R}$. Now wave your hands, do a little dance, and invoke the Axiom of Choice to get a bijection between $\mathbb{R}$ and an ordinal, say $f : \lambda \to \mathbb{R}$. Notationally, we might write $f(\alpha) = r_\alpha$ and think of $f(\alpha)$ as "the α^{th} element of $\mathbb{R}$." Then we can think of this as an enumeration

$$\mathbb{R} = \{r_\alpha\}_{\alpha\in\lambda} = \{r_0, r_1, r_2, \ldots, r_\omega, r_{\omega+1}, \ldots, r_\alpha, \ldots\}$$

All of the ordinals we've explicitly discussed so far in this class have been countable. That's troubling if we want to say that some ordinal is in bijection with $\mathbb{R}$! Let's convince ourselves that uncountable ordinals exist.

Proposition 3.2. *ω_1, the set[1] of all countable ordinals is an uncountable ordinal.*

This jives with our intiution for constructing ordinals. After all, an ordinal is just the set of all ordinals smaller than it.

Remark 3.3. Tying back to Fact 5.2, ω_1 is a fixed point of any increasing continuous function that takes countable ordinals to countable ordinals.

Proof. Assume $\alpha \in \beta \in \omega_1$. By the definition of ω_1, β is countable. By transitivity, $\alpha \subseteq \beta$, so α is also countable. Then by the definition of ω_1, $\alpha \in \omega_1$. We conclude that ω_1 is transitive and therefore an ordinal.

Now assume that ω_1 is countable. Then $\omega_1 \in \omega_1$; contradiction. □

Now let's take on faith that there exist ordinals of every cardinality. We'll use that for this really neat fact:

Theorem 3.4. *There is a set of points $P \subseteq \mathbb{R}^2$ in the plane such that each line intersects P exactly twice.*

If we try to define it explicitly, we'll get frustrated pretty quickly. Systematically lying down points in the plane such that no three of them are collinear is hard! In fact, our construction will depend on Choice, so there isn't a meaningful sense in which we can write an explicit definition.

Proof. We'll build P by induction. First, let λ be the smallest ordinal with cardinality equal to the cardinality of $\mathbb{R}$. Next, let's enumerate the set of lines in $\mathbb{R}$ as $L = \{l_\alpha\}_{\alpha\in\lambda}$. For the base case, let P_0 be some point on l_0.

[1] You might be skeptical that this collection of ordinals forms a set. If so, you should ask Steve about that.

Successor case Inductively assume that P_β intersects each line in L at most twice and that it intersects each line in $L_\beta = \{l_\alpha\}_{\alpha\in\beta}$ exactly twice. If $P_\beta \cap l_\beta$ is two points, let $P_{\beta+1} = P_\beta$. Otherwise, if $P_\beta \cap l_\beta$ is a single point, let x be a point on l_β that is not collinear with any pair of points already in P_β. If $P_\beta \cap l_\beta$ is empty, let x and y be two such points. (We'll prove this is always possible, but let's finish the construction first.) Define $P_{\beta+1} = P_\beta \cup \{x\}$ or $P_{\beta+1} = P_\beta \cup \{x, y\}$ as appropriate.

We want to prove that the inductive hypothesis holds for $P_{\beta+1}$. This is immediate: we didn't add any point to any lines that already had two, so every line still has at most two. The lines that already had exactly two still do, and now l_β is one of them.

Now why is it always possible to find these points? Since $\beta < \lambda$ and λ is the least ordinal of its cardinality, the cardinality of β is smaller than the cardinality of λ. Each pair of points in P_β defines a unique line which intersects l_β in at most one place. Since we add finitely many points at each stage of our induction, the cardinality of P_β is less than the cardinality of λ, so the set of disallowed points can't cover a whole line (which has cardinality λ).

Limit case Inductively assume that for each $\gamma < \beta$, P_γ intersects each line in L at most twice and that it intersects each line in L_γ exactly twice. Define $P_\beta = \bigcup_{\gamma\in\beta} P_\gamma$. To see that for each $\gamma < \beta$, P_β intersects l_γ at most twice, consider $P_{\gamma+1}$. This intersects l_γ exactly twice, and none of the $P_{\gamma'}$ for $\gamma' > \gamma+1$ will change that.

Assume that there is some line l_δ such that $l_\delta \cap P_\beta$ contains p_1, p_2, p_3. Each of these points must be in some P_γ, say $P_{\gamma_1}, P_{\gamma_2}, P_{\gamma_3}$. Let $\gamma = \max\{\gamma_1, \gamma_2, \gamma_3\}$. Then P_γ breaks the inductive hypothesis; contradiction. We conclude that P_β intersects each line in L at most twice.

We conclude that P_β satisfies the inductive hypothesis.

Notice that the successors case relied on the assumption $\beta < \lambda$, but that the limit case didn't. Then $P_\lambda = \bigcup_{\gamma<\lambda} P_\gamma$ also satisfies the inductive hypothesis: for each $\gamma \in \lambda$, P_λ intersects l_γ exactly twice. We're done — these are all of the lines in L!

$\square$

4 Exercises

It should be true that all of the proof techniques required for these exercises have been shown in class. I recommend you try to solve all of them — at the very least, play with the computations in exercise 4.2

Exercise 4.1. Prove that ordinal multiplication is associative, ie. $(\alpha\beta)\gamma = \alpha(\beta\gamma)$. (Hint: induct on γ and apply continuity in the limit case.)

Exercise 4.2. Simplify each of these expressions as much as you can. (This is intentionally vague.) (Hint: Use continuity.)

$(1+1)\omega$	2^ω	$\omega(1+1)$	$(\omega^\omega)^2$	$\omega\omega^\omega$
$1^{\omega 2}$	$\omega(1+1)$	$0+\alpha$	$0 \cdot (\omega^2+\omega 2+1)$	$\omega+\omega^2+\omega^3 3$

Exercise 4.3. Which of the following are true for arbitrary α, β, γ? Provide proofs or counterexamples. (Hint: Exercise 4.2 should build your intuition.)

If any of the statements are false, are they true when restricted to smaller classes of ordinals, eg. finite, limit, countable?

- $(\alpha+\beta)\gamma = \alpha\gamma+\beta\gamma$
- $\alpha(\beta+\gamma) = \alpha\beta+\alpha\gamma$
- $0+\alpha = \alpha$
- $0\alpha = 0$
- $0^\alpha = 0$
- $(\alpha^\beta)^\gamma = \alpha^{\beta\gamma}$

Can you think of any other plausible-sounding arithmetic relations? Are they true or false?

Exercise 4.4. Prove that ω^α is a limit ordinal for any α.

Exercise 4.5. Prove that there exists a set X of real numbers such that the every real number is the distance between exactly one pair of points in X. (Hint: Let λ be the least ordinal with cardinality equal to $\mathbb{R}$. Enumerate $\mathbb{R} = \{r_\alpha\}_{\alpha<\lambda}$. Inductively define X_γ for $\gamma < \lambda$ so that X_γ satsfies the unique distance property for the set $\mathbb{R}_\gamma = \{r_\alpha \,|\, \alpha < \gamma\}$. Check that X_λ is the desired set.)

5 Problems

All of the definitions that you should need in order to read and appreciate the statements of these problems (except for maybe ‘string’ or ‘uncountable’) have been given in class.

Problem 5.1. Explore conditions on α, β such that $\alpha\beta = \beta\alpha$. We say that α and β commute. (If you’ve thought about this for a while, come to me and I’ll give you necessary and sufficient conditions, which you can try to prove if you like. Warning: the proof is hard!) To start, consider these questions: does ω commute with $\omega + 2$? $\omega 2$? ω^2? ω^ω?

Warning: the next few problems require a little bit of set theory background.

Fact 5.2. *If f is continuous and weakly increasing, ie. $\alpha \leq f(\alpha)$ for all α, then f has arbitrarily large fixed points. That is, for any α, there are $\beta > \alpha$ such that $f(\beta) = \beta$.*

Proof. Consider the sequence $X = \left\{\alpha, f(\alpha), f(f(\alpha)), f^3(\alpha), \ldots\right\}$. Let $\beta = \sup X$. If X has a maximum, then $\beta = f^n(\alpha) = f^{n+1}(\alpha)$ for some n, so $\beta = f(\beta)$. Otherwise, β is a limit ordinal, so by continuity, $f(\beta) = \bigcup_{n\in\mathbb{N}} f(f^n(\beta)) = \bigcup_{n\in\mathbb{N}} f^n(\beta) = \beta$. $\square$

Problem 5.3. Prove that if f is continuous, weakly increasing, and takes countable ordinals to countable ordinals, then f has uncountably many fixed points below ω_1. (Hint: a countable union of countable sets is countable.)

Let’s talk about writing down ordinals. Let $A = \{(,), +, \cdot, \hat{}, \omega, 0, 1, 2, \ldots\}$ be the alphabet consisting of the arithmetic operations, the finite ordinals, and ω. We can write strings like $\omega\hat{}(w \cdot 2) + (\omega\hat{}2) \cdot 5 + 17$ in this language and interpet them in the obvious way (so that the string we just wrote is interpreted as $\omega^{w2} + \omega^2 5 + 17$). If we let ε_0 be the smallest ordinal satisfying $\omega^{\varepsilon_0} = \varepsilon_0$, then we have no way to write this down in our language. More generally:

Problem 5.4. Prove that you can’t write down all of the countable ordinals. More precisely, there is no countable alphabet A together with an interpretation of finite strings of symbols from A such that every countable ordinal is the interpretation of some string.

Ordinal Arithmetic Day 3

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July 25, 2015

4 Hessenberg Arithmetic

On last night's homework, I asked you to "simplify these ordinal arithmetic expressions as much as possible". Now I'll give you a precise sense in which an ordinal can be in simplest form:

Theorem 4.1 (Cantor Normal Form)**.** *If α is an ordinal, then there is a unique decreasing sequence of ordinals β_i and a unique finite sequence of finite ordinals n_i such that*

$$\alpha = \sum_i \omega^{\alpha_i} n_i$$

In order to prove this, we'll need the following notion:

Definition 4.2. The *degree* of α, denoted $\deg \alpha$, is the greatest δ such that $\omega^\delta \leq \alpha$.

Proposition 4.3. *If α is any ordinal,* $\deg \alpha$ *exists.*

Proof. Let β be the least ordinal such that $\alpha < \omega^\beta$. Assume that β is a limit ordinal. Then $\omega^\beta = \bigcup_{\gamma<\beta} \omega^\gamma$. Since $\alpha \in \omega^\beta$, there is $\gamma < \beta$ such that $\alpha \in \omega^\gamma$. This contradicts the minimality of β. Therefore, we conclude that $\beta = \delta + 1$ is a successor ordinal. Then $\deg \alpha = \delta$. □

Proof of Theorem 4.1. Let α be an ordinal and $\delta = \deg \alpha$. We have $\omega^\delta \leq \alpha < \omega^{\delta+1} = \omega^\delta \omega$. Therefore, there is some $n \in \omega$ such that $\alpha < \omega^\delta n$. Then there is a maximal k such that $\omega^\delta k \leq \alpha$. Furthermore, there is a unique β such that $\omega^\delta k + \beta = \alpha$. (Proof: exercise)

Now we proceed by transfinite induction. Assume that for all $\alpha' < \alpha$, α' has a Cantor Normal Form. Then there are unique k, β such that $\alpha = \omega^\delta k + \beta$. $\beta < \omega^\delta \leq \alpha$, so by the inductive hypothesis, β has a Cantor Normal Form $\sum_i \omega^{\beta_i} n_i$. Therefore, $\alpha = \omega^\delta k + \sum_i \omega^{\alpha_i} n_i$ is the Cantor Normal Form for α. □

Now we'll define new arithmetic operations on ordinals that pretend Cantor Normal Forms are just polynomials in ω. This makes sense of the term "degree", which now corresponds precisely to the degree of a polynomial!

Definition 4.4 (Hessenberg addition)**.** If $\alpha = \sum_i \omega^{\gamma_i} a_i$ and $\beta = \sum_i \omega^{\gamma_i} b_i$, then

$$\alpha \oplus \beta = \sum_i \omega^{\gamma_i}(a_i + b_i).$$

Right away we can see that this is different from normal addition, for it is commutative. To see that it is more than just an abelianization of normal addition, notice

$$(\omega + 2) + (\omega + 2) = \omega 2 + 2 \neq \omega 2 + 4 = (\omega + 2) \oplus (\omega + 2).$$

We won't prove it, but here is an order-theoretic interpretation of Hessenberg addition. Recall that the *order type* of a well-ordered set is the unique ordinal with which it is order-isomorphic. $\alpha \oplus \beta$ is the supremum over the order types of all wosets extending the poset $\alpha \sqcup \beta = \{0\} \times \alpha \cup \{1\} \times \beta$ with the union order: $(n, \gamma) \prec (m, \gamma')$ iff $n = m$ and $\gamma < \gamma'$.

Definition 4.5 (Hessenberg multiplication)**.** If $\alpha = \sum_i \omega^{\alpha_i} a_i$ and $\beta = \sum_i \omega^{\beta_i} b_i$, then

$$\alpha \otimes \beta = \bigoplus_{i,j} \omega^{\alpha_i \oplus \beta_j}(a_i + b_j).$$

Just as before, we can see immediately that this operation is commutative, and we can compute

$$(\omega+2)(\omega+2) = (\omega+2)\omega+(\omega+2)2 = \omega^2+\omega 2+2 \neq \omega^2+\omega 4+4 = (\omega+2)\otimes(\omega+2)$$

Order-theoretically, $\alpha \otimes \beta$ is the supremum over the order types of all wosets extending the poset $\alpha \times \beta = \{(\gamma, \delta) \mid \gamma \in \alpha, \delta \in \beta\}$ with the product order: $(\gamma, \delta) < (\gamma', \delta')$ iff $\gamma < \gamma'$ and $\delta < \delta'$.

It should be clear that Hessenberg multiplication distributes over Hessenberg addition. In fact, these operations are about as algebraically nice as we could hope for — the ordinals with these operations are a commutative semiring which embeds into the Surreal Numbers.[1]

It's also important to note that $\oplus$ and $\otimes$ agree with $+$ and $\cdot$ when the second argument is finite.

It is easily seen that Hessenberg addition is *not* continuous:

$$\bigcup_{n \in \omega} 2 \oplus \omega = \bigcup_{n \in \omega} 2 \oplus n = \omega \neq 2 \oplus \omega$$

and neither is Hessenberg multiplication:

$$\bigcup_{n \in \omega} n \otimes (\omega + 2) = \bigcup_{n \in \omega} \omega n + 2n = \omega^2 \neq \omega \otimes (\omega + 2)$$

So remember our goal from the class ad: we want to find an exponentiation operation that fits together nicely with Hessenberg addition/multiplication. Here's a proposal:

Definition 4.6 (Super-Jacobsthal Exponentation)**.**

- $\alpha^{\otimes 0} = 1$
- $\alpha^{\otimes(\beta+1)} = \alpha^{\otimes\beta} \otimes \alpha$
- If β a limit, $\alpha^{\otimes\beta} = \bigcup_{\gamma\in\beta} \alpha^{\otimes\gamma}$

Unexpectedly, this has at least one nice property:

Theorem 4.7 (Altman, 2015[2])**.** $\alpha^{\otimes(\beta\oplus\gamma)} = \alpha^{\otimes\beta} \otimes \alpha^{\otimes\gamma}$

But you say "That looks just like everything else we've proven in this class — I wouldn't call that unexpected at all! " Woah, hold on there. Consider $\alpha^{\beta+\gamma} = \alpha^\beta\alpha^\gamma$. One way to interpret this is to say that multiplying α by itself $\beta+\gamma$ times is the same as multiplying it by itself β times and then γ times. Using the same intuition, we should expect

$$\alpha^{\otimes(\beta+\gamma)} = \alpha^{\otimes\beta} \otimes \alpha^{\otimes\gamma}, \tag{1}$$

since $\otimes$-ing α with itself $\beta+\gamma$ times is the same as $\oplus$-ing α with itself β times and then γ times. What does it even mean to do an operation $\beta\oplus\gamma$ many times? Should we do something like look at all possible sequences of partial Hessenberg products of β many 'α's and γ many 'α'sand take the maximum? What?

Before you think we've stated the wrong theorem, notice that equation 1 is just false:

$$\omega^{\otimes(1+\omega)} = \omega^{\otimes\omega} \neq \omega^{\otimes 1} \otimes \omega^{\otimes\omega} = \omega^{\otimes(\omega+1)}$$

Now that we have a vague idea that this theorem is unusual, let's try to prove it. Our standard argument is to just bash it with transfinite induction, using continuity in the limit case. However, we seem to run into a bit of a problem — $\otimes$ is not continuous! Recall that $\cdot$ is continuous in the right argument but not the left, which forces an unpleasant dichotomy: multiplication-like operations can either be commutative or continuous, but not both. Then how should we proceed with the proof?

Well, the proof would actually take about a whole class period to cover. Furthermore, it isn't super enlightening. It goes something like:

First, make a technical observation about how Super-J exponentiation operates on Cantor Normal Forms. Next, do some symbolic bashing to show that both sides of the equation are equal. Unlike some of the proofs we did yesterday, there isn't any known order-theoretic interpretation.

In any case, Theorem 4.7 is true, which is a good sign for Super-J exponentiation to be the operation we're after. Here's another property we clearly want it to have:

Proposition 4.8.

$$\alpha^{\otimes(\beta\otimes\gamma)} = (\alpha^{\otimes\beta})^{\otimes\gamma}$$

Like last time, we shouldn't expect to have any good techniques to prove it, since transfinite induction will fail us miserably. But that's okay, since we have a clean disproof:

Counterexample. Let's make an intermediate calculation:

$$(\omega+1)^{\otimes\omega} = \bigcup(\omega+1)^{\otimes n} = \bigcup \omega^n + \omega^{n-1}n + \ldots = \omega^\omega$$

With this in hand, we shall show

$$((\omega+1)^{\otimes(\omega+1)})^{\otimes 2} \neq (\omega+1)^{\otimes\omega 2+2}$$

$$\begin{array}{rr}
((\omega+1)^{\otimes(\omega+1)})^{\otimes 2} & (\omega+1)^{\otimes\omega 2+2} \\
((\omega+1)^{\otimes\omega} \otimes (\omega+1))^{\otimes 2} & (\omega+1)^{\otimes 2} \otimes (\omega+1)^{\otimes\omega 2} \\
(\omega^\omega \otimes (\omega+1))^{\otimes 2} & (\omega+1)^{\otimes 2} \otimes \displaystyle\bigcup_{n\in\omega} (\omega+1)^{\otimes(\omega+n)} \\
(\omega^{\omega\oplus 1} + \omega^\omega)^{\otimes 2} & (\omega+1)^{\otimes 2} \otimes \displaystyle\bigcup_{n\in\omega} (\omega+1)^{\otimes\omega} \otimes (\omega+1)^{\otimes n} \\
(\omega^{\omega+1} + \omega^\omega)^{\otimes 2} & (\omega+1)^{\otimes 2} \otimes \displaystyle\bigcup_{n\in\omega} \omega^\omega \otimes (\omega+1)^{\otimes n} \\
(\omega^{\omega+1} + \omega^\omega)^{\otimes 2} & (\omega+1)^{\otimes 2} \otimes \displaystyle\bigcup_{n\in\omega} \omega^{\omega+n} \oplus \omega^{\omega+n-1}n \oplus \ldots \\
(\omega^{\omega+1} + \omega^\omega)^{\otimes 2} & (\omega+1)^{\otimes 2} \otimes \omega^{\omega 2}
\end{array}$$

The left side has four nonzero terms and the right side has three. □

Definition 4.9 (Jacobsthal Multiplication)**.**

- $\alpha \times 0 = 0$
- $\alpha \times (\beta+1) = \alpha \times \beta \oplus \alpha$
- If β a limit, $\alpha \times \beta = \bigcup_{\gamma\in\beta} \alpha \times \gamma$

Fact 4.10. *Degree satisfies the following relations: (for the exponentiation ones, assume $\alpha \geq \omega$)*

$$\begin{array}{ccc}
\deg(\alpha+\beta) = \max(\deg\alpha, \deg\beta) & \deg(\alpha\beta) = \deg\alpha + \deg\beta & \deg\alpha^\beta = (\deg\alpha)\beta \\
 & \deg(\alpha\otimes\beta) = \deg\alpha \oplus \deg\beta & \deg\alpha^{\otimes\beta} = (\deg\alpha)\times\beta \\
 & & \boxed{\phantom{\deg\alpha^{\otimes\beta} = (\deg\alpha)\times\beta}}
\end{array}$$

Where $\times$ is transfinitely iterated $\oplus$.

What goes in the box? Well, it would be really, really pretty if we had some "Hessenberg exponential" function $e(\alpha,\beta)$ there so that $\deg e(\alpha,\beta) = (\deg\alpha)\otimes\beta$. This frames our last result:

Theorem 4.11 (Altman, 2015[2]). *There is no function e such that:*

1. *For all α, $e(\alpha, 1) = \alpha$.*
2. *For $\alpha > 1$ or $\beta > 0$, $e(\alpha, \beta)$ is weakly increasing in both arguments*
3. *For all α, β, γ, $e(\alpha, \beta \oplus \gamma) = e(\alpha, \beta) \otimes e(\alpha, \gamma)$.*
4. *For all α, β, γ, $e(\alpha, \beta \otimes \gamma) = e(e(\alpha, \beta), \gamma)$.*
5. *For $\alpha \geq \omega$, $\deg e(\alpha, \beta) = (\deg \alpha) \otimes \beta$*

This will be long and technical and totally unlike everything in this class so far. In particular, we won't make any use of transfinite induction — we're proving something about the class of ordinal *functions*, not something about the class of ordinals. (Spolier: we're going to use a property of the real numbers!)

Lemma 4.12. *If such a function e exists, then there is a function $f : \mathbb{N} \to \mathbb{N}$ with the following three properties:*

- *$f(n^k) = kf(n)$,*
- *f is weakly increasing,*
- *For $n \geq 2$, $f(n) \geq 1$. (In particular, f is not identically 0)*

Proof of Theorem, assuming Lemma. By the properties of logarithms, we have

$$m^{\lfloor \log_m n \rfloor} \leq n \leq m^{\lceil \log_m n \rceil}.$$

f is weakly increasing, so we can transform this to get

$$\lfloor \log_m n \rfloor f(m) \leq f(n) \leq \lceil \log_m n \rceil f(m).$$

Doing some division,

$$\left\lfloor \frac{\log n}{\log m} \right\rfloor \leq \frac{f(n)}{f(m)} \leq \left\lceil \frac{\log n}{\log m} \right\rceil.$$

Make the substitution $n \mapsto n^k$:

$$\left\lfloor k \frac{\log n}{\log m} \right\rfloor \leq k \frac{f(n)}{f(m)} \leq \left\lceil k \frac{\log n}{\log m} \right\rceil.$$

We can weaken this inequality to

$$k \frac{\log n}{\log m} - 1 \leq k \frac{f(n)}{f(m)} \leq k \frac{\log n}{\log m} - 1,$$

and then divide by k to get

$$\frac{\log n}{\log m} - \frac{1}{k} \leq \frac{f(n)}{f(m)} \leq \frac{\log n}{\log m} - \frac{1}{k}.$$

Letting $k \to \infty$,

$$\frac{f(n)}{f(m)} = \frac{\log n}{\log m}$$

The left hand side is always rational, but we may choose m and n so that the right hand side is not. Contradiction! $\square$

Proof of Lemma. The desired function will be $f(n) = \deg e(n, \omega)$. We need to establish both that this function takes naturals to naturals (a priori it might map to infinite ordinals) and that it satsfies the relations in the lemma. Recalling the definition of degree, we know that $\deg \alpha < \omega$ iff $\alpha < \omega^\omega$. So we wish to establish $e(n, \omega) < \omega^\omega$. Let's get some facts about e as applied to finite ordinals.

If $k > 0$ is finite, then by hypotheses (1) and (3),

$$e(\alpha, k) = \alpha^{\oplus k}.$$

Then if $n \geq 2$, we have by hypothesis (2) that for all $k \in \omega$, $e(n, \omega) \geq e(n, k) = n^k$. So $e(n, \omega) \geq \omega$.

Applying hypothesis 4 and the above, we have for finite n, k:

$$e(n^k, \alpha) = e(e(n, k), \alpha) = e(n, k \otimes \alpha) = e(n, \alpha \otimes k) = e(e(n, \alpha), k) = e(n, \alpha)^{\otimes k}$$

Now assume there is some n so that $e(n, \omega) \geq \omega^\omega$. Applying the above, we calculate

$$e(n^2, \omega) = e(n, \omega)^{\otimes 2} \geq (\omega^\omega)^{\otimes 2} = \omega^{\omega 2} > \omega^{\omega + 1}.$$

But this is too big! By hypothesis (5), $\deg e(\omega, \omega) = \deg \omega \otimes \omega = \omega$. By definition of degree, $\omega^\omega \leq e(\omega, \omega) < \omega^{\omega+1}$, contradicting hypothesis (2).

We've established $e(n, \omega) < \omega^\omega$, so that $f(n) = \deg e(n, \omega)$ is in fact a map $\mathbb{N} \to \mathbb{N}$. It is weakly increasing because both degree and e are. $f(n^k) = kf(n)$ by hypothesis (5). Finally, if $n \geq 2$, then $e(n, \omega) \geq \omega$, so $\deg e(n, \omega) \geq 1$. $\square$

5 Exercises

Fill in the details in the proofs! Here's the only detail that's obviously missing for me:

Exercise 5.1. If $\alpha < \beta$, there is a (unique) γ such that $\alpha + \gamma = \beta$. (For intuition, think about this order-theoretically.)

References

[1] John Conway. *On Numbers and Games.* Academic Press, Inc., 1976

[2] Harry Altman *Transfinitely Iterated Natural Multiplication of Ordinal Numbers.* 23 Jan 2015, http://arxiv.org/abs/1501.05747v4