Wizards vs. Time Machines

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Outline

1. What is Complexity Theory?
   - Models of computation
   - Complexity classes
   - Interactive Proofs
   - Closed Timelike Curves
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Fix some finite alphabet $\Sigma$. Let $\Sigma^*$ be the set of finite strings with characters from $\Sigma$. A decision problem or language $L$ is a subset of $\Sigma^*$.
Decision problems, II

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For example, let $\Sigma = \{(,)\,0,1\}$ and let $\text{MATRIX MULTIPLICATION}$ be the set of strings $(A,B,C)$ for which each of $A, B, C$ is an $n^2$-length list of binary strings, and $AB = C$ when these are interpreted as $n \times n$ matrices of binary integers.
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- Informally, we say that a process decides $L$ if it has some behavior for $x \in L$ and a different behavior for $x \notin L$.
- A model of computation is a way to specify what kind of thing the computational process can be.
What is Complexity Theory?

A precise model of computation

Definition (Turing Machine)

A Turing machine $T$ consists of...

(Photo from http://www.aturingmachine.com/)
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- An infinite tape for symbols to sit on
- A “head” which points to some square on the tape.
- A finite list of instructions, one for each element of $\Gamma \times \Sigma$

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Computation *halts* when the machine enters the *accept state* or the *reject state*.
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Polynomial time

Definition

Decidability We say that a language $L$ is \textit{decidable} by Turing machine $T$ if

- When $T$ is run on $x \in L$, $T$ halts and accepts,
- When $T$ is run on $x \notin L$, $T$ halts and rejects.

Let $f : \mathbb{N} \to \mathbb{N}$. We say that $T$ decides $L$ \textit{in time} $f$ if $T$ decides $L$ when run on an input of length at most $n$, $T$ halts within $f(n)$ steps.

Definition $P$ We say that $L \in P$ or $L$ is \textit{decidable in polynomial time} if there is some polynomial $p$ and Turing machine $T$ such that $T$ decides $L$ in time $p$.
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Our definition of Turing machine is arbitrary. The class of polynomials is closed under multiplication and composition. So P is closed under subroutines and poly-length for-loops. In particular, any two sufficiently powerful models of a computer can simulate each other in polynomial time. P would be the same if we replace our Turing machine with a multi-tape Turing machine, a Python program, DNA-based computation, etc.
A problem in P

Example

MATRIXMULTIPLICATION is in P.
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Proof.

The standard matrix multiplication algorithm for two $n \times n$ matrices takes about $n^3$ arithmetic operations. Implement this algorithm in your favorite programming language.
P captures the notion of “solvable in a reasonable amount of time on a normal computer”. For our purposes, we will consider polytime computations as a “baseline” upon which everything else rests.
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Beyond P

P captures the notion of “solvable in a reasonable amount of time on a normal computer”. For our purposes, we will consider polytime computations as a “baseline” upon which everything else rests. In the rest of the talk, we’ll discuss different ways to augment the power of polytime Turing machines by providing additional resources.
Randomness as a resource

Randomness is a useful resource!
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**Definition**

We say $L \in \text{BPP}$ if there is a deterministic polynomial time algorithm $M$ such that when $r$ is chosen uniformly at random,

- If $x \in L$, then $M(x, r)$ accepts with probability at least $\frac{2}{3}$.

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A problem for which randomness helps.

**Definition**

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- $x^4 + y^4 + (x + y)^4$ is identically 0,
- while $x^3 + y^3 + (x + y)^3 = x^2 y + xy^2$ is not identically 0
A randomized algorithm

Lemma (Schwartz-Zippel)

Let $p = p(x_1, x_2, \ldots, x_n)$ be a polynomial of degree $d$ over a field $F$. 

If a nonzero polynomial has degree which is small compared to the size of the field, then a random point is not a zero with high probability. This suggests a BPP algorithm for PIT: pick a random point and evaluate the polynomial. If it’s a zero, declare that the polynomial is zero. If not, declare that it’s not. (If the degree is not small compared to the field, enlarge the field by moving to a field extension.)

Doing the same with a brute-force search would take exponential time.
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One-way, one-round proof system

You want to solve a decision problem. You show the problem to Merlin and he gives you a piece of advice.
One-way, one-round proof system

You want to solve a decision problem. You show the problem to Merlin and he gives you a piece of advice. You don’t trust him, so you have to check it yourself.

Definition (NP)

A language $L$ is in NP if there is a poly time algorithm $V$ (the verifier) such that

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We’ll refer to \( w \) variously as a witness, proof, or certificate.
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You want to solve a decision problem. You start by generating a question which you ask to Merlin. Merlin gives you an answer, and then you use his answer to come to a decision.

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You have two graphs, $G$ and $H$, which you suspect are isomorphic. You want to prove this with Merlin’s help. You undertake the following protocol:
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If $G \not\sim H$, then Merlin can always distinguish.
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- Randomly permute the graph you picked; hand it to Merlin.
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If $G \not\cong H$, then Merlin can always distinguish.
If $G \cong H$, then the situation is identical from Merlin’s point of view, regardless of which graph you picked. He will be right with probability exactly $\frac{1}{2}$. 

1 Jalex Stark

Wizards vs. Time Machines
**AM**

**Definition**

We say a language is in AM if there is a randomized poly-time algorithm $A$, a function $M$, and a verifier $V$ such that when $x$ is an input of length at most $n$,

- If $x \in L$, then $V(x, A(x), M(x, A(x)))$ accepts with probability at least $1 - \varepsilon$

We require that the gap between $\varepsilon$ and $\eta$ is at least $1/p(n)$ for some polynomial $p$. 

---

Fact: $\text{NP} \subseteq \text{AM}$.

Fact: Graph isomorphism is in $\text{NP} \cap \text{coMA}$. 

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$\text{NP} \subseteq \text{AM}$.

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We require that the gap between \( \varepsilon \) and \( \eta \) is at least \( 1/p(n) \) for some polynomial \( p \).

**Fact**

\( \text{NP} \subseteq \text{AM} \).

NP and AM are not necessarily closed under complement.

Jalex Stark

Wizards vs. Time Machines
What is Complexity Theory?

Models of computation
Complexity classes
Interactive Proofs
Closed Timelike Curves

PSPACE

Space is more valuable than time!
PSPACE

Space is more valuable than time!

Definition

We say that a Turing machine \( T \) decides \( L \) in space \( s \) if \( T \) decides \( L \) and whenever \( T \) is run on an input of length at most \( n \), it touches only \( s(n) \) squares on its tape.
Space is more valuable than time!

**Definition**

We say that a Turing machine $T$ decides $L$ in space $s$ if $T$ decides $L$ and whenever $T$ is run on an input of length at most $n$, it touches only $s(n)$ squares on its tape. We say that a language $L$ is in PSPACE if there is a polynomial $p$ and a Turing machine deciding $L$ in space $p$. 
What is Complexity Theory?

PSPACE

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Theorem (PSPACE is big)

1. $P \subseteq NP \subseteq PSPACE$
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**Theorem (PSPACE is big)**
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2. $P \subseteq coNP \subseteq PSPACE$
3. $P \subseteq BPP \subseteq PSPACE$
Proof that PSPACE is big, I

Idea: Polynomial space is big enough to do \textit{brute-force search}.

\textbf{Proof of }\mathsf{NP} \subseteq \mathsf{PSPACE}.

Suppose that \( L \in \mathsf{NP} \). Let \( V \) be a verifier and let \( q \) be a polynomial such that for \( x \in L \) of length at most \( n \), there is a witness \( w \) of length at most \( q(n) \).
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Suppose that $L \in \text{NP}$. Let $V$ be a verifier and let $q$ be a polynomial such that for $x \in L$ of length at most $n$, there is a witness $w$ of length at most $q(n)$.

We describe a PSPACE-algorithm deciding $L$.

1. Initialize $w \leftarrow 0^{q(n)}$. 

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1. Initialize \( w \leftarrow 0^{q(n)} \).

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Interactive Proofs
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Models of computation
Complexity classes
Interactive Proofs
Closed Timelike Curves

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5. Otherwise, increment \( w \) and return to step 2.
Proof that PSPACE is big, II

Lemma

PSPACE is closed under complement.
Lemma

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This establishes that $\text{NP} \subseteq \text{PSPACE}$ iff $\text{coNP} \subseteq \text{PSPACE}$. 
Proof that PSPACE is big, II

Lemma

PSPACE is closed under complement.

This establishes that NP ⊆ PSPACE iff coNP ⊆ PSPACE.

Proof.

Given a PSPACE-algorithm for problem $L$, switch the accept and reject states. This is a PSPACE-algorithm for $\overline{L}$. □
Proof that PSPACE is big, III

In our previous brute force search, we only cared about finding a single point in the search space with a specified property. PSPACE-computations can do much more than that, however.

Proof.

Proof that $\text{BPP} \subseteq \text{PSPACE}$ Let $L \in \text{BPP}$ with algorithm $M$ such that $\Pr_r[M(x, r) \text{accepts}] \geq \frac{2}{3}$ for $x \in L$ and $\Pr_r[M(x, r) \text{accepts}] \leq \frac{1}{3}$ for $x \notin L$. We give a PSPACE-algorithm deciding $M$:

1. Initialize $r$ to the all-zeroes string. Initialize counters “accept” and “reject”.

In fact, this shows that $\text{PP} \subseteq \text{PSPACE}$, where $\text{PP}$ is the class of problems solvable by randomized algorithms without the requirement that there be a large separation between the accept and reject probabilities.
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3. If \( r \) is not the maximum value, increment \( r \) and return to step 2.
4. If the accept counter is larger, halt and accept. Otherwise,
What is Complexity Theory?

Models of computation
Complexity classes
Interactive Proofs
Closed Timelike Curves

Outline

1. What is Complexity Theory?
   - Models of computation
   - Complexity classes
   - Interactive Proofs
   - Closed Timelike Curves
In 1949, Kurt Gödel proved that the equations of general relativity allow for the existence of *closed timelike curves*. 
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In 1949, Kurt Gödel proved that the equations of general relativity allow for the existence of closed timelike curves. These are regions of spacetime where you can travel in a spatial loop and end up back at the beginning \textit{before} you started. How can we use these to do computation?
The grandfather paradox

Here is a “proof” that interaction with CTCs is impossible.

- Travel along the CTC until you come out 50 years before you enter.
The grandfather paradox

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- Travel along the CTC until you come out 50 years before you enter.
- Shoot your grandparent in the head, killing them.
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- Fail to be born.
- Fail to step into the CTC.
Here is a “proof” that interaction with CTCs is impossible.

- Travel along the CTC until you come out 50 years before you enter.
- Shoot your grandparent in the head, killing them.
- Fail to be born.
- Fail to step into the CTC.
- Contradiction!
The Markov chain model

Consider two states of the universe, corresponding to whether or not you are alive.
The Markov chain model

Consider two states of the universe, corresponding to whether or not you are alive. Let each of these states be a basis element in a two-dimensional vector space:

\[
\begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ is “you are alive” and } \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ is “you are dead”}.
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Consider the “perform the shoot-my-grandparent experiment” operator which interchanges these:

\[
\begin{pmatrix}
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\begin{pmatrix}
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\end{pmatrix}
= \begin{pmatrix}
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1
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The apparent contradiction arises because we want this experiment to *not* change the state of the world.
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This contradiction is easily resolved: set the state of the world as

\[
\frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\]
Stable distributions of Markov chains

Say that a finite-dimensional matrix $A$ is a Markov chain if:

- It takes probability distributions to probability distributions.
Stable distributions of Markov chains

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**Theorem**

*If $A$ is a Markov chain, then $A$ has a unique $+1$-eigenvalue eigenvector, called its stable distribution.*
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**Theorem**

*If $A$ is a Markov chain, then $A$ has a unique $+1$-eigenvalue eigenvector, called its stable distribution. Furthermore, it has the following explicit form:*

$$
\nu = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} A^i \nu_0, \quad (1)
$$

*for any starting distribution $\nu_0$.***
A $\text{BPP}_{\text{CTC}}$ algorithm for NP

**Definition**

A language $L$ is in $\text{BPP}_{\text{CTC}}$ if it can be decided in polynomial time by a randomized algorithm which can find stable distributions of implicitly-defined Markov chains as a unit time operation.
A $\text{BPP}_{CTC}$ algorithm for NP

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Markov chains can encode brute force searches in much the same way as PSPACE algorithms.
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Markov chains can encode brute force searches in much the same way as \( \text{PSPACE} \) algorithms. Suppose we have \( L \in \text{NP} \) with verifier \( V \) with an input of length \( n \). Let there be one basis state \( |w\rangle \) for each possible witness \( w \), along with one extra “accept basis state” \( |\text{accept}\rangle \).

Let \( A|w\rangle = |w+1\rangle \) if \( V(x, w) \) rejects and \( A|w\rangle = |\text{accept}\rangle \) if \( V(x, w) \) accepts. Let \( A|\text{accept}\rangle = |\text{accept}\rangle \).
What is Complexity Theory?

Wizards = Time Machines

\[ QIP \overset{[Jai+10]}{=} IP \]
Wizards = Time Machines

\[ QIP \overset{[\text{Jai}+10]}{=} IP \overset{[\text{Sha}92]}{=} PSPACE \]
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Models of computation
Complexity classes
Interactive Proofs
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\[ QIP \overset{\text{[Jai+10]}}{=} IP \overset{\text{[Sha92]}}{=} PSPACE \overset{\text{[AW09]}}{=} BPP_{CTC} = BQP_{CTC} \] (2)
Bibliography I

