Ma191 Geometrical Paradoxes

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1 Paradoxes of Infinity

Galileo's *Dialogues concerning two new sciences* (1638) discusses $f(n) = n^2$ as a bijection between a set and a proper subset.

Dedekind's *Essays on the Theory of Numbers, Continuity of Irrational Numbers, the Nature and Meaning of Numbers* (1888) proves the following:

Theorem 1.1 (Dedekind)**.** *A set is infinite iff it can be put into bijection with a proper subset of itself.*

Theorem 1.2. *If X is an infinite set, there is a partition $X = X_1 \sqcup X_2$ so that X is in bijection with both X_1 and X_2.*

Definition 1.3. If $A, B \subseteq \mathbb{R}^n$, then we say $A \cong B$ or A is *congruent* to B iff there is some distance-preserving function $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\varphi(A) = B$.

Example 1.4. $\mathbb{N} \cong \mathbb{N}^+$ via $n \mapsto n + 1$. Notice that both of these sets are unbounded.

Example 1.5. In two dimensions, we can do this with bounded sets. Consider complex numbers as $\mathbb{R}^2$. Let $|c| = 1$, $c = e^{i\theta}$ with θ not a rational multiple of π. Then if $A = \{c^n : n \in \mathbb{N}\}$, $B = \{c^n : n \in \mathbb{N}^+\}$, we have $B = cA$ so that $B \cong A$.

Remark 1.6. If $A \subseteq \mathbb{R}$ is bounded and $B \subseteq A$ is congruent to A, then $B = A$.

Proof. Any distance-preserving function on $\mathbb{R}$ is of the form $\varphi(x) = \pm x + a$ for some $a \in \mathbb{R}$. In the positive case, notice that the least upper bound of A is distinct from the least upper bound of $\varphi(A)$, so that they must be different. In the negative case, notice that $\varphi^2(x) = x$. Then $\varphi(A) = B \subseteq A$. $\varphi(B) \subseteq \varphi(A)$ since $B \subseteq A$. So $A = \varphi(B) \subseteq \varphi(A) = B$. In other words, $A \subseteq B$, so $A = B$. ☐

Definition 1.7. $A, B \subseteq \mathbb{R}^n$. We say A, B are *equidecomposable* if $A = \bigsqcup_{i=1}^k A_i, B = \bigsqcup_{i=1}^k B_i$ and $A_i \cong B_i$ for all $i \leq k$. We write $A \sim B$.

Remark 1.8. $[0, 1] \sim (0, 1]$

Proof. Let $\{x\} = x - \lfloor x \rfloor$ be the positive fractional part of x. Let $\alpha \in (0,1)$ be irrational. Let $A = \{\{n\alpha\} : n \in \mathbb{N}\}$ and $B = \{\{n\alpha\} : n \in \mathbb{N}^+\}$, so that $A = B \cup \{0\}$. We will show $A \sim B$.

$A_1 = A \cap [0, 1-\alpha)$, $A_2 = A \cap (1-\alpha, 1)$. Notice that if $1-\alpha \in A$, then α is rational, so $1 - \alpha \notin A$.

$B_1 = B \cap [\alpha, 1)$, $B_2 = B \cap (0, \alpha)$. We have $B_1 = A_1 + \alpha$ and $B_2 = A_2 + (1-\alpha)$.

Now we put $A_3 = B_3 = [0,1] \setminus A = (0,1] \setminus B$. Since $[0,1] = A_1 \sqcup A_2 \sqcup A_3$ and $(0,1] = B_1 \sqcup B_2 \sqcup B_3$, we're done. □

Example 1.9 (Mazurkiewicz–Sierpinski). There is a countable $A \subseteq \mathbb{R}^2$ and a partition $A = A_1 \sqcup A_2$ such that $A \cong A_1 \cong A_2$.

Theorem 1.10. *No bounded set can have the above property.*

Definition 1.11. $A \subseteq \mathbb{R}^2$ is called *paradoxical* is there exists a partition $A = A_1 \sqcup A_2$ such that $A \sim A_1 \sim A_2$.

Theorem 1.12 (Banach (1923))**.** *No "reasonable" subset of* $\mathbb{R}^2$ *is paradoxical. For example, bounded sets with nonempty interior are not paradoxical.*

There is a finitely additive isometry-invariant extension of Lebesgue measure defined on all subsets of $\mathbb{R}^2$. That is, there is a function $\mu : \mathcal{P}(\mathbb{R}^2) \to [0, \infty]$ such that $\mu(A) = \lambda(A)$, (where λ is the Lebesgue measure) and $\mu(A \cup B) = \mu(A) + \mu(B)$ if $A \cap B = \emptyset$ and $\mu(\varphi(A)) = \mu(A)$. Then if $A \sim B$, $\mu(A) = \mu(B)$. Then if A is paradoxical, $\mu(A) = 2\mu(A) \in \{0, \infty\}$.

Theorem 1.13 (Sierpinsky (1946))**.** *There are no nonempty paradoxical sets in* $\mathbb{R}^1$*.*

Theorem 1.14 (Banach–Tarski Paradox (1924))**.** *Any solid ball in* $\mathbb{R}^3$ *is paradoxical. Equivalently, any two bounded sets in* $\mathbb{R}^3$ *with non-empty interior are equidecomposable.*

Theorem 1.15 (Tarski Circle Squaring Problem (Laczkovich))**.** *A square and a circle of equal area in the plane are equidecomposable.*

Theorem 1.16 (Marczeaski Problem (Dougherty–Foreman))**.** *Banach–Tarski can be done with pieces that have the property of Baire.*

Theorem 1.17 (Trevor Wilson)**.** *Banach–Tarski can be done "physically": with pieces moving continuously and staying disjoint.*

2 Isometries in Euclidean Space

Definition 2.1. A map $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is an *isometry* if $|\varphi(x) - \varphi(y)| = |x - y|$ for all $x, y \in \mathbb{R}^n$. Recall A, B are *congruent* if there is an isometry φ so that $\varphi(A) = B$.

Example 2.2. $\varphi(x) = x + a$, $a \in \mathbb{R}^n$ is an isometry.

Example 2.3. Rotation in $\mathbb{R}^2$ around the origin by θ is a linear isometry. The matrix with repsect to the standard basis is

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Proposition 2.4. *If $L : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry with $L(0) = 0$, then L is linear.*

Proof. First, see that $x \cdot y = L(x) \cdot L(y)$. This follows from the polarization identity $x \cdot y = \frac{1}{2}\left(|x|^2 + |y|^2 - |x-y|^2\right)$. So we have

$$L(x) \cdot L(y) = \frac{1}{2}\left(|L(x)-0|^2 + |L(y)-0|^2 - |L(x)-L(y)|^2\right)$$
$$L(x) \cdot L(y) = \frac{1}{2}\left(|L(x)-L(0)|^2 + |L(y)-L(0)|^2 - |L(x)-L(y)|^2\right)$$
$$L(x) \cdot L(y) = \frac{1}{2}\left(|x-0|^2 + |y-0|^2 - |x-y|^2\right)$$
$$L(x) \cdot L(y) = x \cdot y$$

Since L preserves dot products, it takes an orthnormal basis to an orthonormal basis, so it is linear.

□

Definition 2.5.

- An *orthogonal transformation* is a linear isometry.
- An $n \times n$ matrix is *orthogonal* if $AA^T = I_n$. In other words, $A^T = A^{-1}$.

We've already shown that for any isometry φ, $\varphi(0) = 0$ iff φ is an orthogonal transformation.

Theorem 2.6. *$L : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal iff its matrix (in the standard basis) is orthogonal.*

Proof. Let L be an orthogonal transformation and A its matrix. Given $x, y \in \mathbb{R}^n$, compute $x^T y = x \cdot y = L(x) \cdot L(y) = (Ax)^T(Ay) = x^T A^t Ay$. Then $x^T(I_n - A^T A)y = 0$, therefore $A^T A = AA^T = I_n$.

Now assume A is orthogonal. Then $L(x) \cdot L(y) = (Ax)^T(Ay) = x^T A^T Ay = x^T y = x \cdot y$. □

Proposition 2.7. *Every isometry of $\mathbb{R}^n$ is a composition of a linear isometry and a translation. In other words, if φ is any isometry, there is an orthogonal transformation L and a constant $c \in \mathbb{R}^n$ so that $\varphi(x) = L(x) + c$. Moreover, L and c are unique.*

Proof. Let $\varphi(0) = c$. Then $L(x) = \varphi(x) - c$ is orthogonal and $\varphi(x) = L(x) + c$. c is uniquely determined from φ and L is uniquely determined from L and c. □

Corollary 2.8. *φ is onto.*

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3 Isometries in Euclidean Space (continued)

Let G_n be the group of all isometries of $\mathbb{R}^n$ and O_n (the *orthogonal group*) be the group of all linear isometries.

If $A \in O_n$, then $\det AA^T = \det I_n$, so $\det A = \pm 1$.

Definition 3.1. $SO_n = \{A \in O_n : \det A = 1\}$ is the subgroup of *direct* or *orientation-preserving* orthogonal transformations. If $\det A = -1$, we say A is *indirect* or or *orientation-reversing*. It's clear that $SO_n \triangleleft O_n$ and $O_n/SO_n \cong \mathbb{Z}/2\mathbb{Z}$.

Let $T_n \cong \langle \mathbb{R}^n, + \rangle$ be the group of translations. Let λ be the homomorphism $L + a \mapsto L$ that sends a transformation to its linear part. Then $T_n = \ker \lambda$ and $O_n = \operatorname{im} \lambda$, so $G_n/T_n \cong O_n$.

We say φ is direct or indirect is $\lambda\varphi$ is direct or indirect, respectively.

Proposition 3.2. *Every direct isometry of $\mathbb{R}^1$ is of the form $\varphi(x) = x + a$. Every indirect isometry is of the form $\varphi(x) = -x + a$.*

Proof. The only 1×1 orthogonal matrices are (1) and (-1). □

Theorem 3.3. *Every direct isometry of $\mathbb{R}^2$ is a translation or rotation around a point.*

Every indirect isometry of $\mathbb{R}^2$ is a glide reflection, ie. a reflection about a line followed by a translation along the line.

Proposition 3.4. *Let A be an orthogonal 2×2 matrix.*

- *If A is direct, then there is some $\theta \in [0, 2\pi)$ so that $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.*
  A corresponds to rotation by θ. As a result, $SO_2 \cong S^1$, the circle group.
- *If A is indirect, then there is some $\theta \in [0, 2\pi)$ so that $A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$.*
  A corresponds to reflection about the line through 0 and $(\cos\frac{\theta}{2}, \sin\frac{\theta}{2})$.

Proof. For orthogonal $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have $a^2+c^2 = 1 = b^2+d^2$, $ab+cd = 0$. So $a = \cos\varphi$, $c = \sin\varphi$ and $b = \sin\psi$, $d = \cos\psi$. $ab + cd = \sin\varphi\cos\psi + \sin\psi\cos\varphi = \sin(\varphi+\psi) = 0$. Therefore, $\varphi+\psi = k\pi$ and $\det A = ad - bc = \cos(k\pi) = \pm 1$.

Case 1 k is even. $\cos\varphi = \cos\psi$ and $\sin\varphi = -\sin\psi$, so $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

A is clearly a rotation by θ.

Case 2 k is odd. $\cos\psi = -\cos\varphi$ and $\sin\psi = \sin\varphi$, so $A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$.

Let $\vec{p} = (\cos\frac{\theta}{2}, \sin\frac{\theta}{2})$, $\vec{q} = (-\sin\frac{\theta}{2}, \cos\frac{\theta}{2})$. Notice $\vec{q} \perp \vec{p}$.

$$\vec{p}A = (\cos\theta\cos\frac{\theta}{2} + \sin\theta\sin\frac{\theta}{2}, \sin\theta\cos\frac{\theta}{2} - \cos\theta\sin\frac{\theta}{2}) = \vec{p}$$

Similarly, $\vec{q}A = -\vec{q}$. Therefore, A is a reflection about the line $\frac{\theta}{2}$. □

Proof of Theorem 3.3. $\varphi(x) = L(x) + \vec{a}$. We'll split into the direct and indirect cases.

Case 1 L is direct, so $L = R_\theta$, rotation about the origin by θ. Let's assume that $\theta \neq 0$, $|\vec{a}| \neq 0$. Find the fixed point x_0 of φ as follows: Consider the perpendicular to a and two lines $\frac{\theta}{2}$ away from it. If x is on the clockwise-side line, then $L(x) = x - t\vec{a}$ for some real t. At the origin, the distance between x and $L(x)$ is 0, and it gets arbitrarily large as we get arbitrarily far away. By the Intermediate Value Theorem, there's some x_0 where $L(x_0) = x_0 - \vec{a}$, so that $\varphi(x_0) = x_0$.

Then φ is rotation about x_0 by the angle θ. If $\theta = 0$, $\vec{a} \neq 0$, φ is a translation. If $\theta = |\vec{a}| = 0$, φ is the identity.

Case 2 L is indirect. Let τ be the line of reflection of L. Let $\vec{a}_\tau^\perp$ and $\vec{a}^\tau$ be the components of $\vec{a}$ perpendicular and parallel to τ, respectively. A simple geometric argument shows that reflecting across τ and then translating by $\vec{a}_\tau^\perp$ is the same as reflecting about the $\frac{1}{2}\vec{a}_\tau^\perp$ translate of τ. Then φ is a glide reflection. □

Definition 3.5 (Isometries in $\mathbb{R}^3$)**.** A *glide-rotation* is a rotation about an axis folllowed by translation in the direction of that axis.

A *reflection-rotation* is a reflection about a plane followed by rotation around the axis orthogonal to the plane.

A *glide-reflection* is a reflection about a plane followed by translation along the axis orthogonal to the plane.

Theorem 3.6.

- *Every direct isometry in $\mathbb{R}^3$ is a glide-rotation.*
- *Every indirect isometry in $\mathbb{R}^3$ is a reflection-rotation or glide-reflection.*

Proposition 3.7. *Every direct orthogonal transformation in $\mathbb{R}^3$ is a rotation around an axis through* 0*.*

Proof. Let A be its matrix. We will show that there is nonzero $\vec{x}$ so that $A\vec{x} = \vec{x}$.

$$\begin{aligned}\det(I_3 - A) &= \det(I_3 - A)\det(A^T)\\ &= \det(A^T - I_3)\\ &= \det((A^T - I_3)^T)\\ &= \det(A - I_3)\\ &= (-1)^3 \det(I_3 - A)\end{aligned}$$

Therefore, $\det(I_3 - A) = 0$, so $(I_3 - A)\vec{x} = 0$ (equivalently $\vec{x} = A\vec{x}$) for some nonzero x.

Let $\vec{i} = \frac{\vec{x}}{\|\vec{x}\|}$. Let $(\vec{i}, \vec{j}, \vec{k})$ be a right-handed orthonormal basis. Let L be the linear part of A. The matrix of L with repsect to this basis is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix}$, with $\det L = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$. (This is determined by $L\vec{i} = \vec{i}$ and the orthogonality of L) Therefore, L is a rotation about the axis determined by $\vec{i}$. □

Proposition 3.8. *If L is an indirect linear isometry in $\mathbb{R}^3$, then L is a reflection followed by a rotation through an axis going through* 0 *and perpendicular to this plane.*

Proof. By the same argument as before, find a unit vector $\vec{i}$ such that $L\vec{i} = -\vec{i}$ and compute that the matrix with respect to an orthnormal basis including $\vec{i}$ as an axis is $\begin{pmatrix} -1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix}$. □

Proof of Theorem 3.6. $\varphi(x) = L(x) + \vec{a}$. We'll split into the direct and indirect cases.

Case 1 L is direct, so it is rotation about the line σ by an angle θ. As in the proof of Theorem 3.3, decompose $\vec{a}$ into $\vec{a}_\sigma^\perp$ and $\vec{a}_\sigma$. We can decompose φ into two parts: $x \mapsto L(x) + \vec{a}_\sigma^\perp$ and $x \mapsto x + \vec{a}_\sigma$. If we restrict to the plane perpendicular to σ at the origin, Theorem 3.3 tells us that our first part is just a rotation about some $\sigma' \parallel \sigma$. Therefore, φ is a glide-rotation. (Notice that if $\theta = 0$, φ is just a translation)

Case 2 L is indirect, so L is a reflection about the plane Π followed by a rotation about σ (orthogonal to Π) of angle θ. If Π' is a translate of Π by $\frac{1}{2}\vec{a}_\sigma$ and σ' is as in Case 1, then φ is a reflection about Π' and a rotation about σ'.

If $\theta = 0$, then φ is a reflection about Π' and a translation by $\vec{a}_\sigma^\perp$. $\square$

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4 Isometries in Euclidean Space (continued)

Definition 4.1. An isometry $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ is *performed continuously* if there is $t \mapsto \varphi_t \in G_3$, $t \in [0,1]$ such that for each $x \in \mathbb{R}^3$, $\varphi_t(x)$ is continuous with $\varphi_0 = \mathrm{id}$ and $\varphi_1 = \varphi$.

Fact 4.2. *φ is performed continuously iff φ is direct.*

Problem 4.3. *L is a direct linear isometry on $\mathbb{R}^3$ with matrix A. Then L is a rotation around an axis through $\vec{0}$. Call the angle of rotation θ. Show that $1 + 2\cos\theta = \operatorname{tr} A$.*

Problem 4.4. *Let* $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$.
Show that A is orthogonal with $\det A = 1$, so corresponds to a rotation around an axis through $\vec{0}$ in the direction of the unit vector $\vec{\eta}$. Calculate $\vec{\eta}$ and the angle of rotation.

5 Measures

Definition 5.1. If X is a set, $\mathcal{P}(X) = \{A : A \subseteq X\}$ is the *power set* of X.
If $X \subseteq A$, then $X \setminus A = A^C$ is the *complement* of A.

Definition 5.2. $\mathcal{A} \subseteq \mathcal{P}(X)$ is an *algebra* if $\emptyset \in \mathcal{A}$ and if $A, B \in \mathcal{A}$, then $A^C, A \cup B \in \mathcal{A}$.
$\mathcal{A}$ is a *σ-algebra* if it also satisfies $\forall n[A_n \in \mathcal{A}] \Rightarrow \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$.

Example 5.3. $\mathcal{A} =$ all finite or co-finite sets is an algebra but not a σ-algebra.

Example 5.4. $\mathcal{S} =$ all countable of co-countable sets is a σ-algebra.

If $C \subseteq \mathcal{P}(x)$, there is a smallest algebra containing C and also a smallest σ-algebra containing C. This is because an intersection of a family of (σ-)algebras is a (σ-)algebra.

Example 5.5. The countable-cocountable σ-algebra is the smallest σ-algebra that contains all singletons.

Definition 5.6. A *finitely additive measure* on an algebra $\mathcal{A} \subseteq \mathcal{P}(X)$ is a map $\mu : \mathcal{A} \to [0, \infty]$ such that $\mu(\emptyset) = 0$ and $\mu(A \cup B) = \mu(A) + \mu(B)$ if $A \cap B = \emptyset$. Here $a + \infty = \infty$ for any $a \in [0, \infty]$.

Consequences of the basic definition:

- $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$. This is because $\mu(B) = \mu(A) + \mu(B \setminus A)$.
- If $\{A_i\}_{i \leq n}$ is a pairwise disjoint family, then $\mu(\bigcup_i^n A_i) = \sum_i^n \mu(A_i)$ by induction.
- $\mu(\bigcup_i^n A_i) \leq \sum_i^n \mu(A_i)$. To see this, let $A'_i = A_i \setminus \bigcup_{k<i} A_k$. Then $A'_i \subseteq A_i$ and $\bigcup_i^n A_i = \bigcup_i^n A'_i$ with the A'_i pairwise disjoint.

Example 5.7. On $\mathcal{A} = \mathcal{P}(X)$, the counting measure is given by

$$\mu(A) = \begin{cases} |A|, & \text{if } |A| < \infty \\ \infty, & \text{otherwise} \end{cases}$$

Example 5.8. On $\mathcal{A}$ the finite-cofinite algebra, the following is a finitely additive measure

$$\mu(A) = \begin{cases} 0, & \text{if } |A| < \infty \\ 1, & \text{otherwise} \end{cases}$$

Definition 5.9. A *countably additive measure* or just *measure* is a finitely additive measure that also has $\mu(\bigcup_{n \in \mathbb{N}} A_n) = \sum_{n \in \mathbb{N}} \mu(A_n)$ for A_n a countable disjoint family of sets.

Example 5.10. Fix $x_0 \in X$. Let

$$\delta_{x_0}(A) = \begin{cases} 0, & \text{if } x_0 \notin A \\ 1, & \text{if } x_0 \in A \end{cases}$$

This is the *Dirac measure on* x_0

Example 5.11. $\mathcal{S}$ is a σ-algebra of countable-cocountable sets.

$$\mu(A) = \begin{cases} 0, & \text{if } A \text{ is countable} \\ 1, & \text{if } A \text{ is cocountable} \end{cases}$$

Example 5.12. Every measure on $\mathcal{P}(\mathbb{N})$ is fully determined by its values on the singletons. In other words, every measure on $\mathcal{P}(\mathbb{N})$ can be representeted by a countable sequence from $[0, \infty]$.

Remark 5.13. There are finitely additive measures μ on $\mathcal{P}(\mathbb{N})$ so that $\mu(\{n\}) = 0$ and $\mu(\mathbb{N}) = 1$.

Problem 5.14. *Let $\mu : \mathcal{S} \to [0, \infty]$ be a measure. Show the following:*

- $A_0 \subseteq A_1 \subseteq \ldots \Rightarrow \mu(\bigcup_n A_n) = \lim_n \mu(A_n)$
- $A_0 \supseteq A_1 \supseteq \ldots \Rightarrow \mu(\bigcap_n A_n) = \lim_n \mu(A_n)$ *if there is at least one* A_i *so that* $\mu(A_i) < \infty$*.*
- $\mu(\bigcup_n A_n) \leq \sum_n \mu(A_n)$

Definition 5.15. A *standard n-box in $\mathbb{R}^n$* is any set of the form $\prod_{i=1}^n (a_i, b_i)$. An *n-box* is a set $\varphi(B)$ for φ an isometry and B a standard n-box.

Definition 5.16. $\mathcal{B}(\mathbb{R}^n)$ is the smallest σ-algebra containing the n-boxes. The sets in $\mathcal{B}(\mathbb{R}^n)$ are called *Borel sets*.

Every open set is Borel, as is every closed set. The collection of Borel sets is equinumerous with the reals.

Theorem 5.17 (Lebesgue)**.** *There is a unique measure $m_n : \mathcal{B}(\mathbb{R}^n) \to [0, \infty]$ with the following two properties:*

- *m_n assigns to each n-box its volume. That is, $m_n(\varphi(\prod_i^n (a_i, b_i))) = \prod_i^n (b_i - a_i)$.*
- *m_n is translation invariant.*

m_n, in general, is called *Lebesgue measure*. If $n = 1, 2, 3$, we sometimes call it length, area, volume, repsectively.

Definition 5.18. A set $A \subseteq \mathbb{R}^n$ is called *null* if there is some $B \supseteq A$ such that $B \in \mathcal{B}(\mathbb{R}^n), m_n(B) = 0$.

Example 5.19.

- Any countable set is null.
- A k-dimensional subspace of $\mathbb{R}^n$ for $k < n$ is null.

Example 5.20. Let $E_0 = [0, 1]$, $E_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, $\ldots$ so that $E = \bigcap_{n \in \mathbb{N}} E_n$ is the Cantor set. Since E_n is closed for all n, E is closed. Furthermore, $E_0 \supseteq E_1 \supseteq \cdots$, so $m_n(E) = \lim_n m_n(E_n) = 0$. Then every subset of the Cantor set is null. Since the Cantor set has cardinality of the reals, it has more subsets than there are Borel sets. Therefore, some subset of the Cantor set is a non-Borel null set.

Definition 5.21. A subset $A \subseteq \mathbb{R}^n$ is *Lebesgue measurable* (LM) if $A = B \cup N$ with B Borel and N null. $\mathcal{LM}(\mathbb{R}^n)$ is the collection of LM sets in $\mathbb{R}^n$.

Proposition 5.22. *A set $A \subseteq \mathbb{R}^n$ is LM iff there exist Borel sets B_1, B_2 such that $B_1 \subseteq A \subseteq B_2$ and $m_n(B_2 \setminus B_1) = 0$*

Proof. Assume A is LM. Then $A = B \cup N$. There is a Borel N' so that $N \subseteq N'$ and N' is null. Then let $B_1 = B$ and $B_2 = B \cup N'$.

Assume $B_1 \subseteq A \subseteq B_2$ with B_1, B_2 Borel and $B_2 \setminus B_1$ null. Let $N = A \setminus B_1$. Then N is null and $A = B_1 \cup N$. □

It follows that $\mathcal{LM}(\mathbb{R}^n)$ is a σ-algebra containing $\mathcal{B}(\mathbb{R}^n)$. Define now $\overline{m}_n : \mathcal{LM}(\mathbb{R}^n) \to [0, \infty]$ by $\overline{m}_n(B \cup N) = m_n(B)$ if N is null.

We have to check that this is well-defined. If $A = B \cup N = B' \cup N'$, then $B' \setminus B \subseteq N$ and $B \setminus B' \subseteq N'$, so $m_n(B) = m_n(B \cap B') = m_n(B')$.

We usually write m_n for $\overline{m}_n$.

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6 Lebesgue Measure

Proposition 6.1. *For any $B \in \mathcal{B}(\mathbb{R}^n), \varphi \in G_n$, we have $\varphi(B) \in \mathcal{B}(\mathbb{R}^n)$ and if $M \in \mathcal{LM}(\mathbb{R}^n)$ then $\varphi(M) \in \mathcal{LM}(\mathbb{R}^n)$ with $m_n(\varphi(M)) = m_n(M)$.*

Proof. Fix an isometry φ. Consider $\mathcal{A} = \{B \in \mathcal{B}(\mathbb{R}^n) : \varphi(B) \in \mathcal{B}(\mathbb{R}^n)\}$. Notice that $\mathcal{A}$ is a σ-algebra since φ preserves complements and unions. Moreover, $\mathcal{A}$ contains all n-balls, so $\mathcal{B}(\mathbb{R}^n) \subseteq \mathcal{A}$. $\mathcal{A} \subseteq \mathcal{B}(\mathbb{R}^n)$ by definition, so $\mathcal{A} = \mathcal{B}(\mathbb{R}^n)$.

Next we prove $m_n(\varphi(B)) = m_n(B)$. Define a measure $m : \mathcal{B}(\mathbb{R}^n) \to [0, \infty]$ by $m(B) = m_n(\varphi(B))$. We'll show that m is translation-invariant and assigns the appropriate volume to n-boxes. Then by the uniqueness of the Lebesgue measure, $m = m_n$, so that $m_n(\varphi(B)) = m_n(B)$.

Since an n-box is an isometric image of a standard n-box, $\varphi(B)$ is an n-box iff B is. Therefore m assigns the appropriate measure to n-boxes.

Let $\varphi(\vec{x}) = L(\vec{x}) + \vec{c}$. Then for any $\vec{a} \in \mathbb{R}^n$,

$$\begin{aligned} m(B + \vec{a}) &= m_n(\varphi(B + \vec{a})) \\ &= m_n(L(B + \vec{a}) + \vec{c}) \\ &= m_n(L(B) + L(\vec{a}) + \vec{c}) \\ &= m_n(L(B) + \vec{c}) \\ &= m_n(\varphi(B)) \\ m(B + \vec{a}) &= m(B) \end{aligned}$$

Therefore m is translation-invariant.

Finally, let $M \in \mathcal{LM}(\mathbb{R}^n)$, so that $M = B \cup N$ with B Borel and N null. Let $C \supseteq N$ with C Borel and $m_n(C) = 0$. Then $\varphi(M) = \varphi(B) \cup \varphi(N)$. $\varphi(N) \subseteq \varphi(C)$, so $\varphi(N)$ is null. Therefore, $\varphi(M) \in \mathcal{LM}(\mathbb{R}^n)$. Furthermore, $m_n(\varphi(M)) = m_n(\varphi(B)) = m_n(B) = m_n(M)$. □

Proposition 6.2. *Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear transformation, so that if A is the matrix of L, $\det A \neq 0$. Then if $B \in \mathcal{B}(\mathbb{R}^n)$, $L(B) \in \mathcal{B}(\mathbb{R}^n)$ and $m_n(L(B)) = |\det A|\, m_n(B)$.*

Proof. Let $\mathcal{S} = \{B \in \mathcal{B}(\mathbb{R}^n) : L(B) \in B(\mathbb{R}^n)\}$. Then $\mathcal{S}$ is a σ-algebra and if B is an n-box, then $L(B)$ is open, so every n-box is in $\mathcal{S}$. Thus $\mathcal{S} = \mathcal{B}(\mathbb{R}^n)$. Let's define a new measure $m(B) = \frac{m_n(L(B))}{|\det A|}$. By linear algebra, m agrees with the Lebesgue measure on the n-boxes.

$$m(B + \vec{a}) = \frac{m_n(L(B + \vec{a}))}{|\det A|} = \frac{m_n(L(B) + L(\vec{a}))}{|\det A|} = \frac{m_n(L(B))}{|\det A|} = m(B)$$

Therefore, m is the Lebesgue measure. □

Theorem 6.3 (Vitali)**.** *There exist sets which are not Lebesgue-measurable.*

Proof. Define an equivalence relation on $[0, 1]$ by $x \sim y \Leftrightarrow x - y \in \mathbb{Q}$. This partitions $[0, 1]$ into countable equivalence classes. Let $V \subseteq [0, 1]$ be a set consisting of exactly one representative from each equivalence class. Assume V is Lebesgue measurable. Then $m_1(V) = m_1(V + q)$. By definition of V, $(V + q) \cap (V + r) = \emptyset$ for any distinct $q, r \in \mathbb{Q}$. Now let $A = \bigcup_{q \in \mathbb{Q} \cap [-1,1]}$. By countable additivity, $m(A) = \sum_{q \in Q \cap [-1,1]} m(V + q)$. Then $m(A) = 0$ if $m(V) = 0$ and $m(A) = \infty$ if $m(V) > 0$. But $[0, 1] \subseteq A \subseteq [-1, 2]$; contradiction. □

Theorem 6.4. *There is a finitely additive measure (f.a.m.) extending the Lebesgue measure that is defined on all of $\mathcal{P}(\mathbb{R}^n)$.*

Definition 6.5. Given a set X, a *ring* on X is a subset $\mathcal{R} \subseteq \mathcal{P}(X)$ such that

- $\emptyset \in \mathcal{R}$
- $A, B \in \mathcal{R} \Rightarrow A \cup B \in \mathcal{R}$
- $A, B \in \mathcal{R} \Rightarrow A \setminus B \in \mathcal{R}$

Notice that the third condition is weaker than being closed under complements, so that every algebra is a ring.

Example 6.6. If X is infinite, then the set of all finite subsets of X is a ring but not an algebra.

Theorem 6.7 (The f.a.m. Extension Theorem)**.** *Let $\mathcal{R}$ be a ring on X and $\mathcal{R} \subseteq \mathcal{A}$ with $\mathcal{A}$ an algebra. Then for any given f.a.m. $\mu_{\mathcal{R}}$ on $\mathcal{R}$ there is a f.a.m. $\mu_{\mathcal{A}}$ on $\mathcal{A}$ such that $\mu_{\mathcal{A}}(R) = \mu_{\mathcal{R}}(R)$ if $R \in \mathcal{R}$.*

Corollary 6.8. *There exists a f.a.m. $\mu : \mathcal{P}(\mathbb{N}) \to [0, 1]$ such that $\mu(\{n\}) = 0$ but $\mu(\mathbb{N}) = 1$.*

Proof. Let $\mathcal{R}$ be the finite-cofinite ring on $\mathbb{N}$. Define

$$\mu_{\mathcal{R}}(A) = \begin{cases} 0, & \text{if } |A| < \infty \\ 1, & \text{if } |A| = \infty \end{cases}$$

and apply Theorem 6.7. □

Definition 6.9. Let $\mathcal{C} \subseteq \mathcal{P}(X)$. Call A an *atom* of $\mathcal{C}$ if $A \neq \emptyset$ and $\mathcal{C} \ni B \subseteq A \Rightarrow B \in \{\emptyset, A\}$.

If $\mathcal{C}$ is finite, define $N(\mathcal{C})$ to be the number of atoms in $\mathcal{C}$.

It is an easy fact that if $\mathcal{C}$ is finite, then for any $A \in \mathcal{C}$, $A \neq \emptyset$, there is an atom $A' \subseteq A$, $A' \in \mathcal{C}$.

Proof of Theorem 6.7, finite case. Assume $\mathcal{A}$ is finite. We'll induct on $N(\mathcal{A})$. If $N(\mathcal{A}) = 1$, then $\mathcal{A} = \{\emptyset, X\}$ and the result is trivial. Now let $N(\mathcal{A}) = n$. If $R = \{\emptyset\}$, then let $\mu_{\mathcal{A}}$ be any measure with $\mu_{\mathcal{A}}(\emptyset) = 0$, say the counting measure.

So assume $\mathcal{R}$ contains a nonempty set. Then $\mathcal{R}$ has an atom R_0. Let $A_0 \subseteq R_0$ with A_0 an atom of $\mathcal{A}$. Let $X' = X \setminus R_0$. Let $\mathcal{A}' = \{A \in \mathcal{A} : A \subseteq X'\}$ and $\mathcal{R}' = \{A \in \mathcal{R} : A \subseteq X'\}$. $\mathcal{A}'$ is an algebra in X' and $\mathcal{R}' \subseteq \mathcal{A}'$ is a ring in X' with $N(\mathcal{A}') < n$. By inductive hypothesis, we have a $\mu_{\mathcal{A}'}$ extending $\mu_{\mathcal{R}} \restriction R'$.

Let $\mathcal{A}^* = \{A \in \mathcal{A} : A \subseteq R_0\}$ and $\mathcal{R}^* = \{A \in \mathcal{R} : A \subseteq R_0\} = \{\emptyset, R_0\}$. Define a measure

$$\mu_{\mathcal{A}^*}(A) = \begin{cases} 0, & \text{if } A \cap A_0 = \emptyset \\ \mu_{\mathcal{R}}(R_0), & \text{if } A \supseteq A_0 \end{cases}$$

Notice that since A_0 is an atom, it must be that $A \cap A_0 \in \{0, A_0\}$. This is a f.a.m. on $\mathcal{A}^*$ extending $\mu_R \restriction \mathcal{R}^*$.

Finally, define $\mu_{\mathcal{A}}(A) = \mu_{\mathcal{A}'}(A \setminus R_0) + \mu_{\mathcal{A}^*}(A \cap R_0)$. Then $\mu_{\mathcal{A}}$ is a f.a.m. extending $\mu_{\mathcal{R}}$. $\square$

Definition 6.10 (Propositional Logic)**.** Fix an indexed set $\{p_i\}_{i \in I}$ of elements called *propositional variables*. Also fix the set of *propositional connectives* and parentheses $\{\neg, \vee, \wedge, \Rightarrow, \Leftrightarrow, (,)\}$. A *formula* is a string of symbols. The class of formulas is the smallest class that contains the propositional variables and is closed under finite application of propositional connectives. That is, the p_i are formulas and if φ, ψ are formulas, then $\neg\varphi, (\varphi \wedge \psi), (\varphi \vee \psi), (\varphi \Rightarrow \psi), (\varphi \Leftrightarrow \psi)$ are formulas.

A *valuation* $v : \{p_i : i \in I\} \to \{0, 1\}$ assigns truth values to variables and can be extended to assign truth values to formulas in the obvious way.

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7 The Compactness Theorem

Definition 7.1. Let $\{p_i\}_{i \in I}$ be an indexed family of propositional variables. A formula is a string in the symbols $\{p_i\}_{i \in I} \cup \{\neg, \vee, \wedge, \Rightarrow, \Leftrightarrow, (,)\}$ defined recursively by:

- Each p_i is a formula
- If ϕ and ψ are formulas, then $(\phi \wedge \psi), (\phi \Rightarrow \psi), \neg\phi, (\phi \vee \psi), (\phi \Leftrightarrow \psi)$ are formulas.

Definition 7.2. A *valuation* is a function $v : \{p_i\}_{i \in I} \to \{0, 1\}$. We extend a valuation to a function on all formulas as follows:

- $v(\neg\phi) = 1 - v(\phi)$
- $v(\phi \wedge \psi) = v(\phi) \cdot v(\psi)$
- $v(\phi \vee \psi) = v(\neg(\neg\phi \wedge \neg\psi))$
- $v(\phi \Rightarrow \psi) = v(\neg\phi \vee \psi)$
- $v(\phi \Leftrightarrow \psi) = v((\phi \Rightarrow \psi) \wedge (\psi \Rightarrow \phi))$

Definition 7.3. If Φ is a set of formulas and v is a valuation, we say v *satisfies* Φ if $v(\phi) = 1$ for all $\phi \in \Phi$. We say that Φ is *satisfiable* if some valuation satisfies it.

Example 7.4. $\Phi_1 = \{p_i, p_j \Rightarrow p_k\}$, $\Phi_2 = \{\neg p_i \wedge (p_j \Rightarrow p_k), p_j \Leftrightarrow p_k\}$, $v(x) = 1$ for all x. Φ_1 is satisfied by v but Φ_2 is not.

Theorem 7.5 (The Compactness Theorem)**.** *Let Φ be any set of formulas, then Φ is satisfiable iff every finite subset of Φ is satisfiable.*

Definition 7.6. A *partial order* on a set P is a binary relation that is reflexive, antisymmetric, and transitive. We call $(P, \leq)$ a *poset.* If we have $m \in P$ so that $m \leq x \Rightarrow x = m$, we call m *maximal.* We call $C \subseteq P$ a *chain* if $\leq \restriction C$ is a linear order.

The following is equivalent to the Axiom of Choice:

Fact 7.7 (Zorn's Lemma)**.** *If P is a nonempty poset such that every chain has an upper bound, then P contains a maximal element.*

Definition 7.8. A *partial valuation* is a valuation defined on any subset of the propositional variables.

Definition 7.9. If f, g are functions we say f *extends* g or $f \supseteq g$ if $\mathrm{dom}\,(g) \subseteq \mathrm{dom}\,(f)$ and $f(x) = g(x)$ for all $x \in \mathrm{dom}\,(g)$.

Proof of 7.5. If v satisfies Φ and $\Phi_0 \subseteq \Phi$, then v satisfies Φ_0. In particular, every finite subset of a satisfiable family is satisfiable.

Now assume that every finite subset of Φ is satisfiable. Let V be the set of all partial valuations such that for any valuation $v \in V$ and finite $\Phi_0 \subseteq \Phi$ there is a valuation $v_0 \supseteq v$ satisfying Φ_0. Order V by extension.

Let C be a chain in V. Clearly, $u = \bigcup C$ is an upper bound for C if $u \in V$. Fix a finite $\Phi_0 \subseteq \Phi$. There are only finitely many variables in Φ_0, so

$$P = \{p_i \in \mathrm{dom}\,(u) \,:\, p_i \text{ appears in } \phi \in \Phi_0\}$$

is finite. Then there is $w \in V$ so that $P \subseteq \mathrm{dom}\,(w)$. Since $w \in V$, there is $v_0 \subseteq w$ satisfying Φ_0. Define

$$v(p_i) = \begin{cases} u(p_i), & \text{if } p_i \in \mathrm{dom}\,(u) \\ v_0(p_i), & \text{if } p_i \notin \mathrm{dom}\,(u) \end{cases}$$

v is a valuation extending u and satisfying Φ_0, so $u \in V$.

By Zorn, let m be a maximal element of V. It suffices to check that m is defined on every propositional variable. Assume toward a contradiction that $q \notin \mathrm{dom}\,(m)$. Extend m to m' by $m'(q) = 0$ if for every finite $\Phi_0 \subseteq \Phi$ there is a valuation $m_0 \subseteq m$ satisfying Φ so $m_0(q) = 0$ and $m_0(q) = 0$ and $m'(q) = 1$ otherwise.

Suppose $m'(q) = 0$. Then $m' \in V$, contradicting maximality of m.

Suppose $m'(q) = 1$. Then there is a finite $\Phi_1 \subseteq \Phi$ so for any $v \subseteq m$ satisfying Φ_1, we have $v(q) = 1$. Fix an arbitrary $\Phi_0 \subseteq \Phi$. There is a valuation $\nu \supseteq m$ satisfiying $\Phi_0 \cup \Phi_1$ and $\nu(q) = 1 = m'(q)$ so $\nu \supseteq m'$ and ν satisfies Φ_0. Therefore, $m' \in V$; contradiction. $\square$

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8 The Extension Theorem

Recall the statement of the extension theorem. We proved the finite case in the Jan 15 lecture.

Theorem 8.1. *Let $\mathcal{R}$ be a ring on X and $\mathcal{A} \supseteq \mathcal{R}$ an algebra. For any f.a.m. on $\mathcal{R}$, we can extend it to a f.a.m. on $\mathcal{A}$.*

Let $\mathcal{R}$ be a finite ring of sets, $\mu_{\mathcal{R}}$ a f.a.m. on $\mathcal{R}$. Let $\mathcal{A}$ be an algebra extending $\mathcal{R}$. For each $A \in \mathcal{A}$ and each $r \in \mathbb{R}$, introduce a propositional variable $p_{A,r}$. We'll think of $p_{A,r}$ as true iff $\mu_{\mathcal{A}}(A) \geq r$. Let Φ contain all formulas of the form:

1. $p_{A,0}$ for all $A \in \mathcal{A}$.
2. $p_{\mathcal{R},r}$ if $R \in \mathcal{R}$ and $\mu_{\mathcal{R}}(R) \geq r$
3. $\neg p_{\mathcal{R},r}$ if $R \in \mathcal{R}$ and $\mu_{\mathcal{R}}(R) < r$
4. $(p_{A,q} \Rightarrow p_{A,r})$ for all $A \in \mathcal{A}$, $q \geq r \in \mathcal{R}$.
5. $(p_{A,r} \Rightarrow p_{B,r})$ for all $A, B \in \mathcal{A}, A \subseteq B$
6. $((p_{A,q} \wedge p_{B,r}) \Rightarrow p_{A\cup B,q+r})$ for all $A, B \in \mathcal{A}, A \cap B = \emptyset$
7. $(\neg p_{A,q} \wedge \neg p_{B,r}) \Rightarrow \neg p_{A\cup B,q+r}$ for all $A, B \in \mathcal{A}, A \cap B = \emptyset$

By Compactness, there is a valuation v satisfying Φ. Define $\mu_{\mathcal{A}}(A) = \sup\{r : v(p_{A,r}) = 1\} \in [0, \infty]$.

Fact 8.2. *If $R \in \mathcal{R}$, then $\mu_{\mathcal{A}}(R) = \mu_{\mathcal{R}}(R)$*

Proof. If $r \leq \mu_{\mathcal{R}}(R)$, then $r \leq \mu_{\mathcal{A}}(R)$ by definition and (2). If $r > \mu_{\mathcal{R}}(R)$, then $\mu_{\mathcal{A}}(R) \leq r$ by (3) and (4), so $\mu_{\mathcal{A}}(R) = \mu_{\mathcal{R}}(R)$. □

Fact 8.3. $\mu_{\mathcal{A}}(A \cup B) = \mu_{\mathcal{A}}(A) + \mu_{\mathcal{A}}(B)$ *if $A \cap B = \emptyset$.*

Proof. If $\mu_{\mathcal{A}}(A) = \infty$ or $\mu_{\mathcal{A}}(B) = \infty$, then $\mu(A \cup B) = \infty$ by (5). Then assume $\mu_{\mathcal{A}}(A) = q, \mu_{\mathcal{A}}(B) = r$. Then $v(p_{A,q-\varepsilon}) = v(p_{B,r-\varepsilon}) = 1$. By (6), $\mu_{\mathcal{A}}(A \cup B) \geq q + r - 2\varepsilon$. Similarly by (7), $\mu_{\mathcal{A}}(A \cup B) \leq q + r + 2\varepsilon$. Since this holds for arbitrary ε, $\mu_{\mathcal{A}}(A \cup B) = q + r$. □

9 Integration on Finitely Additive Measures

Let X be any set and $\mu : \mathcal{P}(X) \to [0, \infty]$ a f.a.m. Fix $A \subseteq X$ with $\mu(A) < \infty$.

Definition 9.1. A map $s : \mathcal{A} \to \mathbb{R}$ is a *step function* if $s = \sum_{k=1}^{n} c_k \chi_{A_k}$ where $A = \bigsqcup_{k=1}^{n} A_k$, $c_k \in \mathbb{R}$, and χ_{A_k} is the characteristic function of A_k.

Define

$$\int_A s \mathrm{d}\mu = \int_A s(x) \mathrm{d}\mu(x) = \sum_{k=1}^{n} c_k \mu(A_k)$$

Example 9.2. Let μ be the counting measure. Then

$$\int_A s \mathrm{d}\mu = \sum_{k=1}^{n} c_k \mu(A_k) = \sum_{x \in A} s(x)$$

Fact 9.3. *If $s = \sum_{j=1}^{m} c_j \chi_{A_j}, t = \sum_{k=1}^{n} c_k \chi_{B_k}$ and $\alpha, \beta \in \mathbb{R}$, then $\alpha s + \beta t$ is a step function and*

$$\int_A (\alpha s + \beta t) \mathrm{d}\mu = \alpha \int_A s \mathrm{d}\mu + \beta \int_A t \mathrm{d}\mu$$

Proof. $\{A_j \cap B_k\}$ is a partition of A and $\alpha s + \beta t$ is constant with value $\alpha c_j + \beta d_k$ on $A_j \cap B_k$ so $\alpha s + \beta t$ is a step function. Then

$$\int_A (\alpha s + \beta t) \mathrm{d}\mu = \sum_{j=1}^{m} \sum_{k=1}^{n} (\alpha c_j + \beta d_k) \mu(A_j \cap B_k) = \alpha \int_A s \mathrm{d}\mu + \beta \int_A t \mathrm{d}\mu$$

The last step follows from $\sum_{k=1}^{n} \mu(A_j \cap B_k) = \mu(A_j)$ and $\sum_{j=1}^{m} \mu(A_j \cap B_k) = \mu(B_k)$. □

Definition 9.4. If $f : A \to \mathbb{R}$ is bounded, ie. $\exists m \exists M \forall x [m \leq f(x) \leq M]$, define the *sup-norm*

$$\|f\|_\infty = \sup_{x \in A} |f(x)| < \infty$$

A sequence of bounded functions $(f_n)_{n \in \mathbb{N}}$ *converges uniformly to* f if $\|f - f_n\|_\infty \to 0$ as $n \to \infty$.

Example 9.5. $A = [0, 1]$. $f(x) = 0$, $f_n(x) = \frac{1}{n} \sin(nx)$. Clearly, $f_n \to f$ uniformly. Let $g_n(x) = nx^n(1 - x)$. Then $g_n(x) \to 0$ for all x, but it is not true that $g_n \to f$ uniformly.

Theorem 9.6. *Suppose $f : A \to \mathbb{R}$ bounded. Then there is a sequence $f_n : A \to \mathbb{R}$ of bounded functions such that $f_n \to f$ uniformly and f_n is a step function.*

Proof. We can assume WLOG $f \geq 0$. Otherwise, f is bounded so $m \leq f$ and we can consider a shift $f - m$. Let $0 \leq f < M$ with $M \in \mathbb{N}$. We'll show that for each n, there is a step function f_n so that $\|f - f_n\|_\infty < \frac{1}{n}$. For each $0 \leq k \leq Mn - 1$, let $I_k = [\frac{k}{n}, \frac{k+1}{n})$. Let $A_k = f^{-1}(I_k)$. Then the A_k partition A, so we can define

$$f_n = \sum_{k=1}^{Mn-1} \frac{k}{n} \chi_{A_k}.$$

Then clearly, $|f_n(x) - f(x)| < \frac{1}{n}$ if $x \in A_k$. Since this holds for all k, we have $\|f_n - f\|_\infty \leq \frac{1}{n}$. □

Definition 9.7. Let $f : A \to \mathbb{R}$ be bounded. Let $f_n \to f$ uniformly with f_n a step function. Then

$$\int_A f \mathrm{d}\mu = \lim_{n \to \infty} \int_A f_n \mathrm{d}\mu$$

Proposition 9.8. *The limit in the above definition exists.*

Proof. $x_n = \int_A f_n \mathrm{d}\mu$. We'll show that $(x_n)_{n \in \mathbb{N}}$ is Cauchy.

$$x_m - x_n = \int_A f_m \mathrm{d}\mu - \int_A f_n \mathrm{d}\mu = \int_A (f_m - f_n) \mathrm{d}\mu$$

We have a bound $|f_m - f_n| \leq \|f_m - f_n\|_\infty$. Applying this bound to every piece of the partition, we have

$$|x_m - x_n| \leq \|f_m - f_n\|_\infty \mu(A) \leq (\|f_m - f\|_\infty + \|f_n - f\|_\infty)\mu(A)$$

The last expression clearly goes to 0 as $n \to \infty$. □

Proposition 9.9. *The limit in the above definition is independent of the choice of f_n.*

Proof. Suppose $f_n \to f$, $g_n \to f$ uniformly with f_n, g_n step functions. Just as before,

$$\left| \int_A f_n \mathrm{d}\mu - \int_A g_n \mathrm{d}\mu \right| \leq \left| \int_A (f_n - g_n) \mathrm{d}\mu \right|$$
$$\left| \int_A f_n \mathrm{d}\mu - \int_A g_n \mathrm{d}\mu \right| \leq \|f_n - g_n\|_\infty \mu(A)$$
$$\left| \int_A f_n \mathrm{d}\mu - \int_A g_n \mathrm{d}\mu \right| \leq (\|f - g_n\|_\infty + \|f_n - f\|_\infty)\mu(A)$$

Where the last expression goes to 0. □

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10 Banach Limits

Definition 10.1. A *Banach limit* L is a function from bounded sequences of real numbers to $\mathbb{R}$ ($\{x_n\} \mapsto L(\{x_n\}) \in \mathbb{R}$) with the following properties:

1. $\forall n[x_n \geq 0] \Rightarrow L(\{x_n\}) \geq 0$
2. $L(\{\alpha x_n + \beta y_n\}) = \alpha L(\{x_n\}) + \beta L(\{y_n\})$
3. $L(\{x_n\}) = L(\{x_{n+1}\})$
4. If $\forall n[x_n = 1]$, $L(\{x_n\}) = 1$.

Fact 10.2. *If L is a Banach limit, then*

$$\underline{\lim}_n x_n \leq L(\{x_n\}) \leq \overline{\lim}_n x_n.$$

In particular, if $\{x_n\}$ converges, then $L(\{x_n\}) = \lim_n x_n$.

Proof. Recall $\overline{\lim}_n x_n = \inf_k \sup \{x_{n+k}\}$. By 3,

$$L(\{x_n\}) = L(\{x_{n+k}\}).$$

Now let $M_k = \sup \{x_{n+k}\}$ so that $x_{n+k} \leq M_k$. By 1,

$$L(\{M_k - x_{n+k}\}) \geq 0.$$

Then by 2,

$$M_k L(\{1\}) - L(\{x_{n+k}\}) \geq 0$$

So by 4,

$$M_k \geq L(\{x_{n+k}\}) = L(\{x_n\}).$$

Therefore, $L(\{x_n\}) \leq \overline{\lim}_n x_n$. The other inequality follows from the same technique. □

Fact 10.3. *There exists a Banach limit.*

Proof. Let $\mu : \mathcal{P}(\mathbb{N}) \to [0,1]$ be a fam such that $\mu(\{n\}) = 0$, $\mu(\mathbb{N}) = 1$. Given a bounded $\{x_n\}$, consider $f : \mathbb{N} \to \mathbb{R}$ given by $f(n) = \frac{1}{n}\sum_{k=0}^{n-1} x_k$. Then let $L(\{x_n\}) = \int_{\mathbb{N}} f \mathrm{d}\mu$. We'll check the Banach limit axioms:

1. If $x_n \geq 0$, $f \geq 0$, so $L(\{x_n\}) = \int_{\mathbb{N}} f \mathrm{d}\mu \geq 0$.
2. Linearity of L follows from linearity of integrals.
4. $L(\{1\}) = \int_{\mathbb{N}} 1 \mathrm{d}\mu = \mu(\mathbb{N}) = 1$
3. Fix $\{x_n\}$ with $|x_n| \leq M$.

$$
\begin{aligned}
I &= L(\{x_n\}) - L(\{x_{n+1}\}) \\
I &= \int_{\mathbb{N}} \left(\frac{1}{n}\sum_{k=0}^{n-1} x_k - \frac{1}{n}\sum_{k=1}^{n} x_k \right) \\
I &= \int_{\mathbb{N}} \frac{x_0 - x_n}{n} \mathrm{d}\mu \\
|I| &= \left| \int_{\mathbb{N}} \frac{x_0 - x_n}{n} \mathrm{d}\mu \right| \\
|I| &= \left| \int_{n \in \mathbb{N} \,:\, n \leq k} \frac{x_0 - x_n}{n} \mathrm{d}\mu + \int_{n \in \mathbb{N} \,:\, n > k} \frac{x_0 - x_n}{n} \mathrm{d}\mu \right| \\
|I| &\leq \left| \int_{n \in \mathbb{N} \,:\, n \leq k} \frac{x_0 - x_n}{n} \mathrm{d}\mu \right| + \left| \int_{n \in \mathbb{N} \,:\, n > k} \frac{x_0 - x_n}{n} \mathrm{d}\mu \right| \\
|I| &\leq 0 + \frac{2M}{k+1} \mu(\{n \in \mathbb{N} \,:\, n > k\}) \\
|I| &\leq \frac{2M}{k+1} \\
|I| &= 0
\end{aligned}
$$

Therefore, $L(\{x_n\}) = L(\{x_{n+1}\})$.

□

11 Group Actions and Equidecomposability

Definition 11.1. Let G be a group and X a set. An *action* of G on X is a map $(g, x) \mapsto g \cdot x \in X$ such that $1 \cdot x = x$ and $g \cdot (h \cdot x) = gh \cdot x$.

Example 11.2. If H is a group and G is a subgroup, then all of the following are actions of G on H:

- $g \cdot x = x$
- $g \cdot x = gx$

- $g \cdot x = xg^{-1}$
- $g \cdot x = gxg^{-1}$

Example 11.3. Suppose G is a permutation group on X. Then $g \cdot x = g(x)$ is a an action.

Example 11.4. $G = (\mathbb{Z}, +)$ and $\sigma : X \to X$ a bijection. Then $n \cdot x = \sigma^n(x)$ is an action.

Definition 11.5. Let G act on X. Define an equivalence relation on X by $xEy \Leftrightarrow \exists g(g \cdot x = y)$. The equivalence classes of this relation are the *orbits* of the action. We say the orbit of x is $[x]_E = G \cdot x = \{g \cdot x : g \in G\}$. G acts *transitively* on X if for all x, $G \cdot x = X$.

X/E is usually written X/G and is called the *orbit space* of the action.

Example 11.6. G_n acting on $\mathbb{R}^n$ by evaluation is transitive.

Example 11.7. If $G \leq H$, with the action $g \cdot x = gx$ then the orbit of x is the right coset Gx.

Example 11.8. Let α be irrational. $\mathbb{Z}$ acts on $\mathbb{T}$ by $n \cdot z = e^{i\pi n\alpha}z$. The orbit of z is the countable set $\left\{e^{i\pi n\alpha}z : n \in \mathbb{N}\right\}$.

Example 11.9. Let $\mathbb{Q}$ act on $\mathbb{R}$ by left-translation. Then the orbit space is the same as the one discussed in the creation of the Vitali set.

Definition 11.10. If G acts on X and $A, B \subseteq X$, we say A and B are *G-congruent* ($A \approx_G B$) if $\exists g \in G[g \cdot A = B]$.

Example 11.11. G_n-congruence on $\mathbb{R}^n$ is exactly usual geometric congruence.

Definition 11.12. A, B are *G-equidecomposable* ($A \sim_G B$) if $A = \bigsqcup_k^n A_k$, $B = \bigsqcup_k^n B_k$, and $A_k \approx_G B_k$.

Example 11.13. For G_n acting on $\mathbb{R}$ we just say equidecomposable and write $A \sim B$.

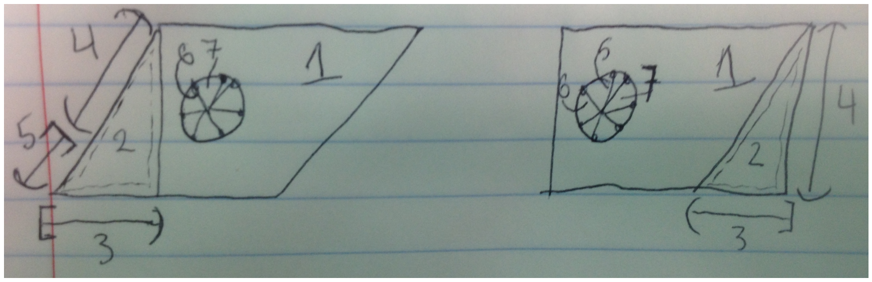
Proposition 11.14. *$\sim_G$ is an equivalence relation.*

Proof. Reflexivity and symmetry are trivial.

Assume $A \sim_G B$ and $B \sim_G C$ by $A_i \xrightarrow{g_i} B_i$ and $B_j \xrightarrow{h_j} C_j$. We can refine the partitions $\{B_i\}$ and $\{B_j\}$ as $\{B_i \cap B_j\}$. This gives rise to partitions $A = \left\{g_i^{-1}(B_i \cap B_j)\right\}$ and $C = \{h_j(B_i \cap B_j)\}$. Then the transitivity of $\sim_G$ follows from the transitivity of $\approx_G$. $\square$

Example 11.15. $[0,1] \not\approx (0,1]$ but $[0,1] \sim (0,1]$ as we showed earlier in the class.

Example 11.16. The parallelogram with width w and height h is equidecomposable with a rectangle of width w and height h. See the diagram below.



Region 1 includes the interior and boundary of the trapezoid minus the circle inside. Region 2 includes only the interior of the triangle. Regions 3, 4, and 5 are intervals on the boundary of the triangle. Region 4 has no endpoints. Regions 3 and 5 each have one endpoint, denoted by the hard bracket. Region 6 (not fully drawn) is a countable union of radii (each including the point on the boundary of the circle but not at the center) generated by an irrational rotation. Region 7 is the circle inside the trapezoid minus regions 5 and 6.

It is obvious how to move all regions but 6 from the left figure to the right figure via rigid motions. Region 6 is rotated forward by one unit in order to vacate the space of the first radius. Region 5 then occupies that space.

Definition 11.17. Let G act on X with $A, B \subseteq X$. We say $A \preceq_G B$ if $A \sim_G B' \subseteq B$.

Theorem 11.18. *$A \preceq_G B$ and $B \preceq_G A$ iff $A \sim_G B$.*

The proof is a modification of a proof of the Cantor–Schroeder–Bernstein theorem, which is a similar statement for set-theoretic injections and bijections.

Ma191 Geometrical Paradoxes

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12 The Banach–Cantor–Schroeder–Bernstein Theorem

Theorem 12.1. *$A \preceq_G B$ and $B \preceq_G A$ iff $A \sim_G B$.*

Proof. If $A \sim_G B$, then there is a bijection $f : A \to B$ such that $A' \sim_G f(A')$ for any $A' \subseteq A$. Additionally, if $A_1 \sim_G B_1$ and $A_2 \sim_G B_2$ with $A_1 \cap A_2 = \emptyset = B_1 \cap B_2$, then $A_1 \cup A_2 \sim_G B_1 \cup B_2$.

Now let $A \sim_G B_1$ by f and $B \sim_G A_1$ by g. Inductively define $C_0 = A \setminus A_1$ and $C_{n+1} = gf(C_n)$. We have $C_n \subseteq A$ and $C_i \cap C_j = \emptyset$ for $i \neq j$. Let $C = \bigcup_n C_n \subseteq A$. Now $A \setminus C \subseteq A_1$. Performing a calculation,

$$g^{-1}(A \setminus C) = g^{-1}(A \setminus \bigcup_n C_n)$$
$$g^{-1}(A \setminus C) = g^{-1}(A) \setminus \bigcup_n g^{-1}(C_n)$$
$$g^{-1}(A \setminus C) = g^{-1}(A) \setminus (\emptyset \cup f(C_0) \cup f(C_1) \cup \cdots)$$
$$g^{-1}(A \setminus C) = g^{-1}(A) \setminus f(C)$$
$$g^{-1}(A \setminus C) = B \setminus f(C)$$

Then we have $C \sim_G f(C)$ and $A \setminus C \sim_g B \setminus f(C)$. Therefore, $A \sim_G B$. $\square$

Definition 12.2. A *similarity* in $\mathbb{R}^n$ is a bijection $S : \mathbb{R}^n \to \mathbb{R}^n$ such that for some $\alpha > 0$, $|S(x) - S(y)| = \alpha |x - y|$ for any $x, y \in \mathbb{R}^n$. Similarities form a group with isometries as a subgroup. We say A and B are *similar* if there is some similarity S with $S(A) = B$.

Fact 12.3. *If $A, B \subseteq \mathbb{R}^n$ have non-empty interior and A, B are bounded, then we can decompose $A = A_1 \sqcup A_2$, $B = B_1 \sqcup B_2$ such that A_1 is similar to B_1 and A_2 is similar to B_2.*

Proof. A *shrinking map* $\alpha(x - x_0) + x_0$ with $0 < \alpha < 1$ and $x_0 \in \mathbb{R}^n$ is a a similarity. Isometries are similarities. By composing a shrinking map and a translation, we can find a similarity S with $S(B) \subseteq A$ and likewise a similarity T with $T(A) \subseteq B$. (Consider that A is contained in a ball of radius M and B contains a ball of radius m; let $0 < \alpha < \frac{m}{M}$) By theorem 12.1, we're done. □

13 Geometric Dissection of Polygons

Definition 13.1. Two polygons are *congruent by dissection* if $P = \bigcup_{i=1}^{n} P_i, Q = \bigcup_{i=1}^{n} Q_i$ such that

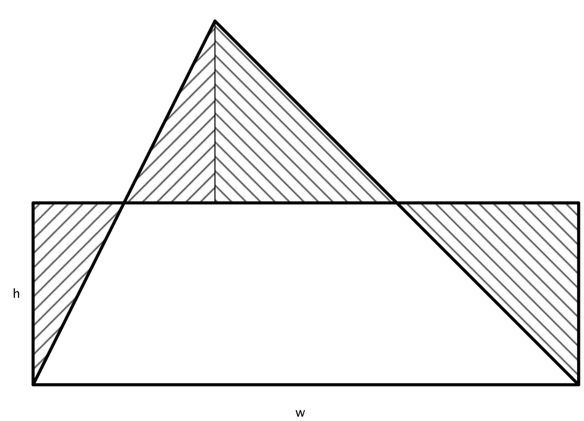
- P_i, Q_i are polygons
- $P_i \approx Q_i$
- P_i, P_j are disjoint ignoring boundaries if $i \neq j$
- Q_i, Q_j are disjoint ignoring boundaries if $i \neq j$

Theorem 13.2 (Bolyai–Gerwien)**.** *Polygons in $\mathbb{R}^2$ are congruent by dissection iff they have the same area.*

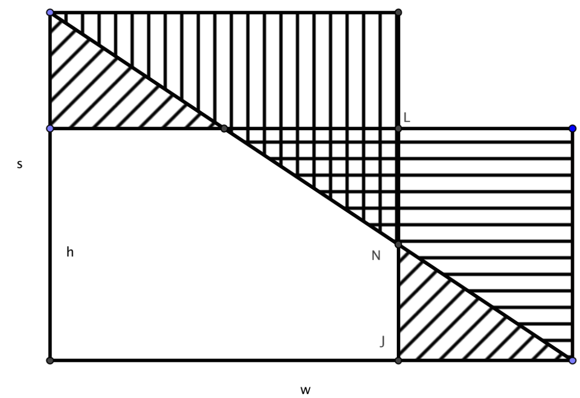
Proof. Obviously, if two polygons are congruent by dissection, they have the same area.

For the other direction, it suffices to show that every polygon is congruent by dissection to a square of the same area.

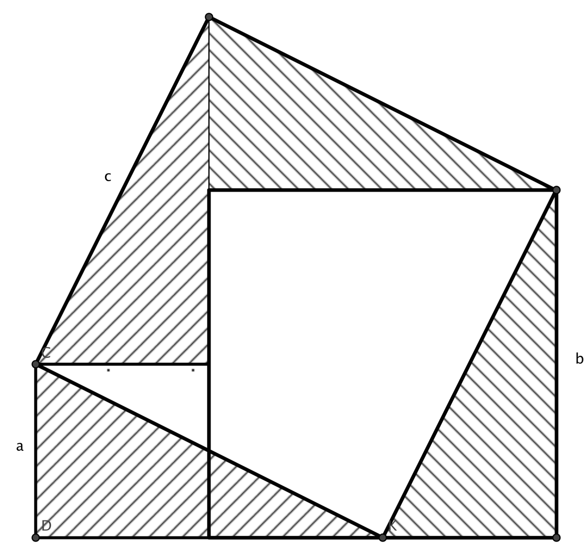
Triangle Case Assume the polygon is a triangle. Then it is congruent by dissection to a rectangle of the same area. Proof by picture:



Now take a rectangle of width w and height h; we'll show it's congruent by dissection to a square with sidelength $s = \sqrt{wh}$. We can assume $h < w < 4h$ (If not, cut the rectangle in half and rearrange the halves to get a rectangle of width $\frac{w}{2}$ and height $2h$.) This guarantees that point N lies on segment LJ in the following diagram.



General Case Decompose the polygon into triangles. This is congruent by dissection to a disjoint union of squares. It suffices to show that the disjoint union of two squares is congruent by dissection to a square. Let the squares have sidelengths a, b and $c = \sqrt{a^2 + b^2}$.



□

Theorem 13.3. *Two polygons are congruent by dissection (have the same area) iff they are equidecomposable.*

We'll show now that congruence by dissection implies equidecomposability. Later, we'll show that equidecomposability implies equality of area.

Definition 13.4. Let $A \subseteq \mathbb{R}^n$. The *interior of* A is

$$A^o = \bigcup_{\substack{G \subseteq A \\ G \text{ open}}} G.$$

The *closure of* A is

$$\overline{A} = \bigcap_{\substack{F \supseteq A \\ F \text{ closed}}} F.$$

The *boundary of* A is $\delta A = \overline{A} \setminus A^o$.

Lemma 13.5. *Suppose $A \subseteq \mathbb{R}^2$ is bounded and has nonempty interior. Let T be a finite union of finite line segments and points such that $A \cap T = \emptyset$. Then $A \sim A \cup T$.*

Proof. Let $D \subseteq A$ be closed disk with radius $r > 0$. Write $T = \bigsqcup_{i=1}^n S_i$ where each S_i is a point, open interval, or half-open interval of length less than r. Use the "shift trick" as in the last lecture to show that $D \cup S_1 \preceq D$. (generate a countable union of line segments via an irrational rotation, then shift if by one rotation to make room for an extra) Obviously $D \preceq D \cup S_1$, so by 12.1 and induction, $A \cup T \sim A$. □

Proof of 13.3. Suppose that $P, Q \subseteq \mathbb{R}^2$ are polygons congruent by dissection, $P = \bigcup_{i=1}^n P_i, Q = \bigcup_{i=1}^n Q_i$ with $P_i \approx Q_i$, $P_i^o \cap P_j^o = \emptyset = Q_i^o \cap Q_j^o$ for $i \neq j$. Then $P_i^o \approx Q_i^o$ and $\bigcup_{i=1}^n P_i^o \sim \bigcup Q_i^o$. Apply the lemma to $A = \bigcup_{i=1}^n P_i^o$ and $T = P \setminus \bigcup_{i=1}^n P_i^o$. Do similarly for Q. Then

$$P \sim \bigcup_{i=1}^n P_i^o \sim \bigcup_{i=1}^n Q_i^o \sim Q$$

□

Definition 13.6. Let G act on X. $A \subseteq X$ is *G-paradoxical* if there are $B, C \subseteq A$, $B \cap C = \emptyset$ and $B \sim_G A \sim_G C$. We call X just paradoxical if it is paradoxical with respect to the group of isometries in $\mathbb{R}^n$. We call G paradoxical if the set G is paradoxical under the action of left-translation by G.

Example 13.7. Let $\mathbb{F}_2$ be the free group on 2 generators, say a and b. We'll show $\mathbb{F}_2$ is paradoxical. For $x \in \mathbb{F}_2$, define $S(x)$ to be the set of all reduced words starting with x. Now $\mathbb{F}_2 = \{1\} \sqcup S(a) \sqcup S(a^{-1}) \sqcup S(b) \sqcup S(b^{-1})$. Words in $S(a^{-1})$ do not have a as their second letter, so $aS(a^{-1}) = \mathbb{F}_2 \setminus S(a)$. Then $S(a) \sqcup S(a^{-1}) \sim_{\mathbb{F}_2} \mathbb{F}_2$. We can do the same with b, so $\mathbb{F}_2$ is paradoxical.

In the above decomposition, 1 was not in either of our pieces. Let's do a different decomposition.

$A_1 = S(a)$, $A_2 = S(a^{-1})$, $A_3 = S(b) \cup \{b^{-n} : n \in \mathbb{N}\}$, $A_4 = S(b^{-1}) \setminus \{b^{-n} : n \in \mathbb{Z}^+\}$. Then we have $A_1 \sqcup aA_2 = \mathbb{F}_2 = A_3 \sqcup bA_4$.

This is a concrete example of a result we'll show in generality later: if X is paradoxical, then we can have $X = A \sqcup B$ with $A \sim_G X \sim_G B$.

Ma191 Geometrical Paradoxes

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14 Properties of Paradoxical Sets

Proposition 14.1.

1. *If A is G-paradoxical and $A \sim_G A'$, then A' is G-paradoxical.*
2. *A set $A \subseteq X$ is G-paradoxical iff there are $B, C \subseteq A$ with $B \cap C = \emptyset$, $B \cup C = A$ such that $A \sim_G B \sim_G C$.*

Proof. (2) The reverse direction is obvious. For the forward direction, let $B, C \subseteq A$ such that $A \sim_G B \sim_G C$ with $B \cap C = \emptyset$. Then $A \preceq_G A \setminus B$ and $A \setminus B \preceq A$. So $A \sim_G A \setminus B = B'$ and $A \sim_G = C'$. Then $A \sim_G B' \sim_G C'$ with $A = B' \sqcup C'$. □

Definition 14.2. Let G act on X and $\mu : \mathcal{P}(X) \to [0, \infty]$ be a fam. Then μ is *G-invariant* if for any $A \subseteq X$ and any $g \in G$, $\mu(g \cdot A) = \mu(A)$.

If μ is G-invariant and $A, B \subseteq X$ are such that $A \sim_G B$, then $\mu(A) = \mu(B)$.

Proposition 14.3. *If $\mu : \mathcal{P}(X) \to [0, \infty]$ is G-invariant and $A \subseteq X$ is such that $0 < \mu(A) < \infty$, then A is not G-paradoxical.*

Proof. Assume not and let $A = B \sqcup C$ with $A \sim_G B \sim_G C$. Then $\mu(A) = \mu(B) = \mu(C)$ and $\mu(A) = \mu(B) + \mu(C)$, so $\mu(A) = 2\mu(A)$. □

A result due to Tarski gives a converse:

Theorem 14.4 (Tarski)**.** *Let G act on X. Then for any $A \subseteq X$, the following are equivalent:*

- *A is not G-paradoxical*
- *There is a G-invariant fam $\mu : \mathcal{P}(X) \to [0, \infty]$ with $\mu(A) = 1$.*

Definition 14.5. Let G act on X and $A \subseteq X$, $f : A \to \mathbb{R}$. Define $f_g : g^{-1} \cdot A \to \mathbb{R}$ by $f_g(x) = f(g \cdot x)$.

Theorem 14.6. *Let G act on X. Let $\mu : \mathcal{P}(X) \to [0, \infty]$ be a G-invariant fam. Let $f : A \to \mathbb{R}$ be bounded, $g \in G$. Then*

$$\int_{g^{-1}\cdot A} f_g \mathrm{d}\mu = \int_A f \mathrm{d}\mu.$$

Proof. First, consider characteristic functions $f = \chi_B$. Then $f_g = \chi_{g^{-1}\cdot B}$, so

$$\int_{g^{-1}\cdot A} f_g \mathrm{d}\mu = \int_{g^{-1}\cdot A} \chi_{g^{-1}\cdot B} \mathrm{d}\mu = \mu(g^{-1} \cdot B) = \mu(B) = \int_A f \mathrm{d}\mu$$

By linearity, the theorem holds for all step functions. Now let $f : A \to \mathbb{R}$ be an arbitrary bounded function and $(f_n)_{n\in\mathbb{N}}$ be a sequence of step functions converging uniformly to f. Then $(f_n)_g$ is a step function on $g^{-1} \cdot A$ and $(f_n)_g \to (f)_g$ uniformly. Then

$$\int_A f \mathrm{d}\mu = \lim_n \int_A f_n \mathrm{d}\mu = \lim_n \int_{g^{-1}\cdot A} (f_n)_g \mathrm{d}\mu = \int_{g^{-1}\cdot A} f \mathrm{d}\mu.$$

□

Fact 14.7. *There are $A, B \subseteq \mathbb{R}^3$ such that $A^o \neq \emptyset$, $B^o = \emptyset$ but $A \sim B$.*

Proof sketch. First, pick a countable dense $D \subseteq S^2$. There are only countably many rotations R so that $D, R(D), R^2(D), \ldots$ are not pairwise disjoint. Since there are uncountably many rotations, pick one so that they are pairwise disjoint. Using the usual shift trick, we see that $S^2 \sim S^2 \setminus D$. We can extend this from S^2 to the unit ball by considering D' the set of radii whose endpoints are in D. □

Definition 14.8. $A \subseteq \mathbb{R}^n$ is *nowhere dense* if $\overline{A}^o = \emptyset$.

Fact 14.9.

- *If $A_1, \ldots, A_n$ are nowhere dense, then $\bigcup_{i=1}^n A_i$ is nowhere dense.*
- *If $A \sim B$ and A is nowhere dense, so is B.*

Proof.

- It suffices to show this for the case $n = 2$. $\overline{A \cup B} = \overline{A} \cup \overline{B}$, so we can assume WLOG that A, B are closed. Assume there is an open nonempty $U \subseteq A \cup B$. Consider $U \setminus A = U \cap (\mathbb{R}^n \setminus A)$. $U \setminus A \subseteq B$, so $U \setminus A = \emptyset$. But then $U \subseteq A$; contradiction.
- Let $A = \bigcup_{i=1}^n A_i, B = \bigcup_{i=1}^n B_i$ with $A_i = g_i B_i$, g_i isometries. $\overline{A_i} \subseteq \overline{A}$, so $\overline{A_i}^o \subseteq \overline{A}^o$

  Since isometries are continuous, (ie. take limit points to limit points) $\overline{A_i} = g_i \overline{B_i}$. Now if $x \in U \subseteq \overline{B_i}$ with U open, then by the continuity of g_i^{-1}, $g_i x \in g_i U \subseteq \overline{B_i}$ with $g_i U$ open. Therefore, $\overline{B_i}^o \subseteq \overline{A_i}^o = \emptyset$. Then B_i is nowhere dense. Since B is a union of nowhere dense sets, it is nowhere dense.

□

15 Paradoxes in Dimension ≤ 2

Definition 15.1. Let G be a group, $S \subseteq G$ finite nonempty, $S = \{g_1, \dots, g_n\}$. Define $B_S(k) = \{g_{i_1} \dots g_{i_l} : l \leq k, 1 \leq i_j \leq n\}$. Let $N_S(k) = |B_S(k)|$.

It is clear that $N_S(k) \leq N_S(k+1)$.

Example 15.2. $G = (\mathbb{Z}, +)$. $S = \{-1, 0, 1\}$. $B_S(k) = \{-k, -k+1, \dots, 0, 1, \dots, k\}$ and $N_S(k) = 2k+1$.

Example 15.3. $G = \mathbb{F}_2 = \langle a, b \rangle$, $S = \{a, b\}$. $B_S(k) = \{x_1 \cdots x_l : l \leq k, x_j \in \{a, b\}\}$ and $N_S(k) = 2^{k+1} - 1$.

Definition 15.4. G has *polynomial growth* if for every finite $S \subseteq G$, there is some $M \in \mathbb{R}^+$, $d \in N$ so that $N_S(k) \leq Mk^d$ for all k.

Remark 15.5. If G is finitely generated, say by S_0, then G has polynomial growth iff there are M, d so that $N_{S_0}(k) \leq Mk^d$. This holds because given any other S', there is some k_0 so that $S \subseteq B_{S_0}(k)$. Then $B_S(k) \subseteq B_{S_0}(kk_0) \leq M(kk_0)^d = (Mk_0^d)k^d$. WLOG, we may assume that S_0 is symmetric, ie. closed under inverses.

Proposition 15.6. *If G is abelian, then G has polynomial growth.*

Proof. Let $S = \{g_1, \dots g_n\} \subseteq G$. Then every element of $B_S(k)$ is of the form $g_1^{a_1} \cdots g_n^{a_n}$ where $a_i \in \{0, \dots, k\}$. So $N_s(k) \leq (k+1)^n \leq 2^n k^n$. □

Theorem 15.7. *G_1, the group of isometries of $\mathbb{R}$, has polynomial growth.*

Proof. $S = \{\varphi_1, \dots, \varphi_n\} \subseteq G_1$. Then $\varphi_i(x) = a_i x + b_i$ with $a_i \in \{-1, 0, 1\}$. Then $\varphi = \varphi_{i_1}\varphi_{i_2} \cdots \varphi_{i_k} = ax + b$ with $a \in \{-1, 0, 1\}$ and $b = \sum_{i=1}^n c_i b_i$ with $c_i \in -k, -k+1, \dots, k$. Then $N_S(k) \leq 3 \cdot (2k+1)^n \leq (2 \cdot 3^n)k^n$. □

Fact 15.8. *$N_S(k+l) \leq N_S(k)N_S(l)$. In particular, $N_S(k) \leq (N_s(1))^k$.*

Proof. Let $g = g_{i_1} \cdots g_{i'_k} \in B_S(k)$ and $h = g_{i_1} \cdots g_{i'_l} \in B_S(l)$ for $k' \leq k$ and $l' \leq l$. Then $gh \in B_S(k+l)$ and the map $B_S(k) \times B_S(l) \to B_S(k+l)$ by $(g, h) \mapsto gh$ is onto. □

Fact 15.9.

$$\alpha = \lim_{k \to \infty} \sqrt[k]{N_S(k)} \text{ exists.}$$

Note that by 15.8, $1 \leq \alpha \leq N_S(1)$

Proof. If $l \geq k$, we have $l \leq k \left\lfloor \frac{l}{k} \right\rfloor + k$, so

$$N_S(l) \leq N_S(k \left\lfloor \frac{l}{k} \right\rfloor + k)$$

$$N_S(l) \leq N_S(k)^{\left\lfloor \frac{l}{k} \right\rfloor} N_s(k)$$

$$N_S(l) \leq N_S(k)^{\frac{l}{k}} N_s(k)$$

$$N_S(l)^{\frac{1}{l}} \leq N_S(k)^{\frac{1}{k}} N_s(k)^{\frac{1}{l}}$$

Let $l \to \infty$, then

$$\overline{\lim}_l N_S(l)^{\frac{1}{l}} \leq N_S(k)^{\frac{1}{k}} \cdot 1.$$

Since k was arbitrary, we have

$$\overline{\lim}_l N_S(l) \leq \underline{\lim}_k N_S(k)^{\frac{1}{k}}.$$

□

Definition 15.10. G is *exponentially bounded* if $\lim_k N_s(k)^{\frac{1}{k}} = 1$ for any finite $S \subseteq G$.

Example 15.11. Any polynomial growth group is exponentially bounded.

Proof. $N_S(k) \leq Mk^d$ so $N_S(k)^{\frac{1}{k}} \leq M^{\frac{1}{k}} k^{\frac{d}{k}}$. Then

$$\lim_{k\to\infty} N_S(k)^{\frac{1}{k}} \leq \left(\lim_{k\to\infty} M^{\frac{1}{k}}\right) \lim_{k\to\infty} (k^{\frac{1}{k}})^d = 1$$

□

Example 15.12. $\mathbb{F}_2 = \langle a, b\rangle$ and $S = \{a, b\}$. $N_S(k) = 2^{k+1} - 1$, so $N_S(k)^{\frac{1}{k}} = (2^{k+1} - 1)^{\frac{1}{k}}$ and $\lim_k N_S(k)^{\frac{1}{k}} = 2 > 1$. Therefore $\mathbb{F}_2$ is not exponentially bounded.

Ma191 Geometrical Paradoxes

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16 Paradoxes in Dimension ≤ 2 (cont.)

Theorem 16.1 (Sierpinski)**.** *If G acts on X and G is exponentially bounded, then there are no non-empty G-paradoxical sets in X.*

Corollary 16.2. *There are no nonempty paradoxical sets in $\mathbb{R}^1$.*

Proof. Suppose $\emptyset \neq A \subseteq X$ is a G-paradoxical set. $A \sim_G B \sim_G C$ with $A = B \sqcup C$. Let $\varphi_1 : A \to B$ and $\varphi_2 : A \to C$ be the bijections given by equidecomposability. Fix $x_0 \in A$ and look at $\varphi_{i_1} \cdots \varphi_{i_k}(x_0)$ with $i_j \in \{1, 2\}$.

Assume $(i_1, \ldots, i_k) \neq (i'_1, \ldots, i'_k)$. Let j be minimal so that $i_j \neq i'_j$. Then $\varphi_{i_j} \cdots \varphi_{i_k}(x_0) \neq \varphi_{i'_j} \cdots \varphi_{i'_k}(x_0)$, since one of these lies in C and the other in B. Then $\varphi_{i_1} \cdots \varphi_{i_{j-1}}$ is injective, so $\varphi_{i_1} \cdots \varphi_{i_k}(x_0) \neq \varphi_{i'_1} \cdots \varphi_{i'_k}(x_0)$.

Let $S = \{g_1, \ldots, g_m, h_1, \ldots, h_n\}$, where the g_i are congruences between the pieces of A and B and h_i are congruences between A and C. Then $\varphi_{i_1} \cdots \varphi_{i_k}(x_0) = f_1 \cdots f_k(x_0)$ with $f_j \in S$. This map is injective, so $N_s(k) \geq 2^k$ and G is not exponentially bounded. $\square$

Proposition 16.3. *If G is not exponentially bounded, then for any $\alpha > 1$, there is a finite $S \subseteq G$ and $n \in \mathbb{N}$ so that $N_s(k) \geq \alpha^k$ if $k > n$.*

Proof. Since G is not exponentially bounded, there is finite $S_0 \subseteq G$ with

$$\lim_{k\to\infty} N_{S_0}(k)^{\frac{1}{k}} = \beta > 1.$$

Let m_0 be such that $\beta^{m_0} > \alpha$. Then $N_{S_0}(m_0 k)^{\frac{1}{m_0 k}} \to \beta$. Then for large enough k, $N_{S_0}(m_0 k) \geq \alpha^k$. Let $S = B_{S_0}(m_0 k)$. Then $B_{S_0}(mk) \subseteq B_S(k)$ and $N_S(k) \geq N_{S_0}(mk) \geq \alpha^k$. $\square$

Theorem 16.4 (Mazurkiewicz–Sierpinski)**.** *There is a countable set $A \neq \emptyset$ in $\mathbb{R}^2$ and a partition $A = A_1 \sqcup A_2$ such that $A \approx A_1 \approx A_2$.*

Proof. Think of $\mathbb{R}^2$ as $\mathbb{C}$. Consider $\mathbb{N}[z]$, the set of polynomials with non-negative integer coefficients. Let $c \in \mathbb{C}$ be transcendental with $|c| = 1$. Let $A = \{p(c) : p \in \mathbb{N}[z]\}$. Let $A_1 = A + 1$ and $A_2 = cA$, so that $A \approx A_1 \approx A_2$. Let $p \in \mathbb{N}[z]$ with $p = \sum_{i=0}^n a_n z^n$. If $a_0 = 0$, then $p = z\sum_{i=1}^n a_n z^{n-1} = zp'$, so $p(c) \in A_1$. Also, $p(c) \notin A_2$, since there is no p' such that $p = p' + 1$. Similarly, if $a_0 \neq 0$, then $p(c) \in A_2$ and $p(c) \notin A_1$. Therefore $A = A_1 \sqcup A_2$. □

Definition 16.5. A group G is *amenable* if there is a G-invariant fam $\mu : \mathcal{P}(G) \to [0,1]$ such that $\mu(G) = 1$.

Recall this theorem of Tarski, which will be proven later:

Theorem 16.6. *If G acts on X and $A \subseteq X$, then A is not G-paradoxical iff there is a G-invariant fam $\mu : \mathcal{P}(X) \to [0,\infty]$ with $\mu(A) = 1$.*

So G is not paradoxical iff G is amenable.

Definition 16.7. G is supramenable if for any nonempty $A \subseteq G$ there is a G-invariant fam $\mu : \mathcal{P}(G) \to [0,\infty]$ with $\mu(A) = 1$.

Applying Tarski's theorem again, G is supramenable iff G has no nonempty paradoxical sets.

Proposition 16.8. *For a group G, we have the following chain of implications:*
G abelian $\Rightarrow$ G has polynomial growth $\Rightarrow$ G is exponentially bounded $\Rightarrow$ G is supramenable $\Rightarrow$ G is amenable.

Proof. We've proven the first two implications already, and the last is obvious. The third follows from 16.1 and 16.6. □

Definition 16.9. A group G is *solvable* if there is a sequence $0 = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_n = G$ such that H_{i+1}/H_i is abelian.

Theorem 16.10. *G_2 is solvable.*

Proof. We claim $\{1\} \trianglelefteq T_2 \trianglelefteq SG_2 \trianglelefteq G_2$, where T_2 is the group of translations and SG_2 is the group of direct isometries.

Obviously, $\{1\} \trianglelefteq T_2$ with $T_2/\{1\} \cong T_2 \cong (\mathbb{R}^2, +)$ abelian.

Given $\varphi \in SG_2$, let φ^* be the corresponding orthogonal transformation. $\varphi \mapsto \varphi^*$ is a surjective homormorphism onto SO_2 whose kernel is T_2. Then $SG_2/T_2 \cong SO_2 \cong (S^1, \cdot)$ is abelian.

Consider the map $\varphi \mapsto \det \varphi^*$. It is a surjective homomorphism from G_2 onto the group of order 2 with kernel SG_2. So $G_2/SG_2 \cong (\{-1,1\}, \cdot)$ is abelian. □

Theorem 16.11. *Every solvable group is amenable.*

Proof. It is enough to check that if $G \trianglelefteq H$ and G and H/G are amenable, then H is amenable. Let μ, ν be translation-invariant fams on $G, H/G$ respectively with $\mu(G) = 1 = \nu(G/H)$. If $C = hG$, define $\mu_C(B) = \mu(h^{-1}B)$. For this to be well-defined, it must be independent of the choice of h. Assume $hG = h_1G$. Then $h_1^{-1}h \in G$, so $\mu(h_1^{-1}B) = \mu((h_1^{-1}h)h^{-1}B) = \mu(h^{-1}B)$.

Given $A \subseteq H$ and $C \in H/G$, let $f_A(C) = \mu_C(A \cap C)$. So let $\lambda(A) = \int_{H/G} f_A \mathrm{d}\nu$. Let's check that λ is a fam.

- $\lambda(\emptyset) = 0$
- $\lambda(H) = 1$ because f_H is the constant 1 function
- $\lambda(A \cup B) = \lambda(A) + \lambda(B)$ when $A \cap B = \emptyset$ because $f_{A\cup B} = f_A + f_B$

Now let's check that λ is H-invariant. Let $A \subseteq H$, $h_0 \in H$. We need to check that $\lambda(h_0A) = \lambda(A)$.

$$\begin{aligned}
f_{h_0A}(C) &= \mu_C(h_0A \cap C) \\
&= \lambda(A) \\
f_{h_0A}(C) &= (f_A)_{h_0^{-1}G}(C) \\
\lambda(h_0A) &= \int_{H/G} f_{h_0A} \mathrm{d}\nu \\
&= \int_{H/G} f_A \mathrm{d}\nu \\
&= \int_{H/G} (f_A)_{h_0^{-1}G} \mathrm{d}\nu \\
&= \lambda(A)
\end{aligned}$$

$\square$

Theorem 16.12. *Let $G \leq G_n$ be amenable. Then there exists a G-invariant fam $\mu : \mathcal{P}(\mathbb{R}^n) \to [0, \infty]$ which extends Lebesgue measure.*

Corollary 16.13. *There are no bounded paradoxical sets in $\mathbb{R}^2$ with nonempty interior.*

Proof of Corollary. G_2 is amenable, so there is an isometry invariant fam μ extending Lebesgue measure. If A is bounded with nonempty interior, then $0 < \mu(A) < \infty$, so $\mu(A) \neq 2\mu(A)$. $\square$

Corollary 16.14. *There is a translation-invariant fam $\mu : \mathcal{P}(\mathbb{R}^n) \to [0, \infty]$ extending Lebesgue measure.*

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17 Paradoxes in $\mathbb{R}^2$

Theorem 17.1. *Let $G \leq G_n$ be amenable. Then there is a G-invariant fam $\mu : \mathcal{P}(\mathbb{R}^n) \to [0, \infty]$ which extends Lebesgue measure.*

Corollary 17.2. *There exists a fam on $\mathcal{P}(\mathbb{R}^2)$ which is isometry-invariant. Therefore, $\mathbb{R}^2$ has no nonempty paradoxical sets that are bounded with nonempty interior.*

Corollary 17.3. *Two polygons in $\mathbb{R}^2$ are equidecomposable iff they are congruent by dissection (ie. have the same area)*

Proof of Corollary 17.3. We've seen in Section 13 that congruence by dissection implies equidecomposability. Let μ be as in 17.2. Then if P, Q are equidecomposable, $\mu(P) = \mu(Q)$. Since polygons are Lebesgue measurable, $m_2(P) = \mu(P) = \mu(Q) = m_2(Q)$. $\square$

Proof of Theorem 17.1. By the fam extension theorem, there is a fam $\nu : \mathcal{P}(\mathbb{R}^n) \to [0, \infty]$ which extends Lebesgue measure. Fix a fam $\lambda : \mathcal{P}(G) \to [0, 1]$ which is left invariant with $\lambda(G) = 1$. For $A \subseteq \mathbb{R}^n$ define $f_A : G \to [0, \infty]$ by $f_A(g) = \nu(g^{-1}A)$. Then define

$$\mu(A) = \begin{cases} \int_G f_A(g) \mathrm{d}\lambda(g), & \text{if } f_A \text{ is bounded} \\ \infty, & \text{otherwise} \end{cases}$$

See that this is a fam extending Lebesgue measure:

- $\mu(\emptyset) = 0$
- $A, B \subseteq \mathbb{R}^n$, $A \cap B = \emptyset$. If either f_A or f_B is unbounded, then so is $f_{A \cup B}$ and $\mu(A \cup B) = \infty = \mu(A) + \mu(B)$. If both are bounded, then $f_{A \cup B} = f_A + f_B$, so $\mu(A \cup B) = \mu(A) + \mu(B)$.
- Let $A \in \mathcal{LM}(\mathbb{R}^n)$. Then $f_A(g) = \nu(g^{-1}A) = m_n(g^{-1}A) = m_n(A)$ so $\mu(A) = \int_G f_A \mathrm{d}\lambda = m_n(A)$.

Now we see that μ is G-invariant. Fix $h \in G$. Then

$$f_{h\cdot A}(g) = \nu(g^{-1}h \cdot A)$$
$$f_{h\cdot A}(g) = f_A(h^{-1}g)$$
$$f_{h\cdot A}(g) = (f_A)_{h^{-1}}(g)$$

So $f_{h\cdot A} = (f_A)_{h^{-1}}$. Then

$$\mu(h \cdot A) = \int_G f_{h\cdot A} \mathrm{d}\lambda$$
$$\mu(h \cdot A) = \int_G (f_A)_{h^{-1}} \mathrm{d}\lambda$$
$$\mu(h \cdot A) = \int_G (f_A) \mathrm{d}\lambda$$
$$\mu(h \cdot A) = \mu(A)$$

□

Theorem 17.4. *Let G act on X.*

(i) If G is amenable, then there is a G-invariant fam on $\mathcal{P}(X)$ such that $\mu(X) = 1$.

(ii) If G is supramenable, then for every nonempty $A \subseteq X$, there exists a G-invariant fam μ on $\mathcal{P}(X)$ such that $\mu(A) = 1$.

Corollary 17.5. *G_2 is not supramenable. (By 16.4, it has a nonempty paradoxical set when acting on $\mathbb{R}^2$, which breaks (ii))*

Proof. We'll just prove (ii); it's clear that (i) follows from the same argument.

Fix $x_0 \in X$ with $G \cdot x_0 \cap A \neq \emptyset$. Let $A_0 = \{g \in G : g \cdot x_0 \in A\}$. Let ν be a fam on $\mathcal{P}(G)$ which is invariant under left-translation and $\nu(A_0) = 1$. Then for $B \subseteq X$, define $\mu(B) = \nu(\{g : g \cdot x_0 \in B\})$. μ is clearly a fam. $\mu(A) = \nu(A) = 1$. Now

$$\mu(h \cdot B) = \nu(\{g : g \cdot x_0 \in h \cdot B\})$$
$$\mu(h \cdot B) = \nu\left(\left\{g : h^{-1}g \cdot x_0 \in B\right\}\right)$$
$$\mu(h \cdot B) = \nu(\{hf : f \cdot x_0 \in B\})$$
$$\mu(h \cdot B) = \nu(h\{f : f \cdot x_0 \in B\})$$
$$\mu(h \cdot B) = \nu(\{f : f \cdot x_0 \in B\})$$
$$\mu(h \cdot B) = \mu(B)$$

□

Theorem 17.6 (Invariant Extension Theorem)**.** *Let G be amenable. Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be G-invariant algebra of sets (ie. $\mathcal{A}$ is closed under the action of G). Suppose $\mathcal{R} \subseteq \mathcal{A}$ is a G-invariant subring of $\mathcal{A}$. Let $\mu_{\mathcal{R}}$ be a G-invariant fam on $\mathcal{R}$. Then there is a G-invariant fam $\mu_{\mathcal{A}}$ on $\mathcal{A}$ extending $\mathcal{R}$.*

Proof. This is a generalization of 17.1, where $\mathcal{R} = \mathcal{LM}(\mathbb{R}^n)$ and $\mathcal{A} = \mathcal{P}(\mathbb{R}^n)$. Let λ be a measure on $\mathcal{P}(G)$ witnessing the amenability of G. Let $\mu_{\mathcal{A}}$ extend $\mu_{\mathcal{R}}$ by the fam extension theorem. Then put $\mu(B) = \int_G \mu_{\mathcal{A}}(g^{-1}B)\mathrm{d}\lambda$. □

18 Amenability

Let's recall the definition:

Definition 18.1. A group G is *amenable* if there is a fam $\mu : \mathcal{P}(G) \to [0,1]$ such that $\mu(G) = 1$ and invariant under left-translation, ie. $\mu(gA) = \mu(A)$ for $g \in G$, $A \subseteq G$.

Example 18.2. Finite groups are amenable with the measure $\mu(A) = \frac{|A|}{|G|}$.

Example 18.3. Consider the group $(\mathbb{Z}, +)$. Let $A \subseteq Z$ and ν a fam on $\mathcal{P}(\mathbb{N})$ with $\nu(\mathbb{N}) = 1$ and $\nu(\{n\}) = 0$. Define

$$\mu(A) = \int_A \frac{|A \cap \{-n, \dots n\}|}{2n+1} \mathrm{d}\nu(n)$$

It's easy to see that μ is a fam. Let's check that $\mu(k + A) = \mu(A)$.

$$|\mu(k+A) - \mu(A)| \leq \int \left| \frac{|(k+A) \cap [-n,n]|}{2n+1} - \frac{|A \cap [-n,n]|}{2n+1} \right| \mathrm{d}\nu(n)$$

$$|\mu(k+A) - \mu(A)| \leq \int \left| \frac{|A \cap [-k-n,-k+n]|}{2n+1} - \frac{|A \cap [-n,n]|}{2n+1} \right| \mathrm{d}\nu(n)$$

$$|\mu(k+A) - \mu(A)| \leq \int \frac{2k}{2n+1} \mathrm{d}\nu(n)$$

$$|\mu(k+A) - \mu(A)| = 0$$

Note that whenever $f : \mathbb{N} \to \mathbb{R}$ is such that $\lim_{n\to\infty} f(n) = 0$, then $\int_{\mathbb{N}} f \mathrm{d}\nu = 0$ because $\int_{\mathbb{N}} f \mathrm{d}\nu = \int_{[0,N]} f \mathrm{d}\nu + \int_{[N,\infty]} f \mathrm{d}\nu$, where the first part is 0 and the second part is $< \varepsilon$ for large enough N.

Proposition 18.4. *A subgroup of an amenable group is amenable.*

Proof. Say H is amenable and $G \leq H$. Consider the right-cosets Gh, $h \in H$, of G in H. Let $S \subseteq H$ contain exactly one element from each coset. Let ν be a H-invariant fam on $\mathcal{P}(H)$ such that $\nu(H) = 1$. Define a fam μ on $\mathcal{P}(G)$ with $\mu(G) = 1$ by $\mu(A) = \nu(AS)$. The fact that μ is a fam follows easily from the fact that ν is a fam. Now

$$\mu(gA) = \nu((gA)S) = \nu(g(AS)) = \nu(AS) = \mu(A)$$

□

Proposition 18.5. *If G is amenable and $N \trianglelefteq G$, then G/N is amenable. Equivalently, if G is amenable and $f : G \to H$ is a surjective homomorphism, then H is amenable.*

Proof. Let $f : G \to H$ be a surjective homomorphism. Let ν be a measure on $\mathcal{P}(G)$ witnessing the amenability of G. Let μ on $\mathcal{P}(H)$ be the push-forward measure $\mu(B) = \nu(f^{-1}(B))$ □

Proposition 18.6.

- *If $N \trianglelefteq G$ with $N, G/N$ amenable, then G is amenable.*
- *If G, H are amenable, then so is $G \times H$.*

Proof. We proved the first statement when proving that solvable groups are amenable (16.11). The second statement is a corollary of the first, as $G \trianglelefteq G \times H$ and $(G \times H)/G \cong H$. □

Proposition 18.7. *If G is amenable, then there is a fam μ on G with $\mu(G) = 1$ and μ is both left- and right-invariant.*

Proof. Let μ_l be a left-invariant fam on G with $\mu_l(G) = 1$. Define $\mu_r(A) = \mu_l(A^{-1})$. Then μ_r is a right-invariant fam on G with $\mu_r(G) = 1$. Now define $\mu : \mathcal{P}(G) \to [0, 1]$ by

$$\mu(A) = \int_G \mu_l(Ag^{-1})\mathrm{d}\mu_r(g)$$

μ is a fam with $\mu(G) = 1$. It is left-invariant by the left-invariance of μ_l:

$$\mu(hA) = \int_G \mu_l(hAg^{-1})\mathrm{d}\mu_r(g) = \int_G \mu_l(Ag^{-1})\mathrm{d}\mu_r(g) = \mu(A)$$

Let $f(g) = \mu_l(Ag^{-1})$; we'll show the right-invariance of μ.

$$\begin{aligned}
\mu(Ah) &= \int_G \mu_l(Ahg^{-1})\mathrm{d}\mu_r(g)\\
\mu(Ah) &= \int_G \mu_l(A(gh^{-1})^{-1})\mathrm{d}\mu_r(g)\\
\mu(Ah) &= \int_G f(gh^{-1})\mathrm{d}\mu_r(g)\\
\mu(Ah) &= \int_G f(g)\mathrm{d}\mu_r(g)\\
\mu(Ah) &= \mu(A)
\end{aligned}$$

where the fourth step comes from the right-invariance of μ_r. □

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19 Amenable Groups

Theorem 19.1. *Let $(G_i)_{i \in I}$ be subgroups of G so that for any finite $S \subseteq G$, there is an $i \in I$ with $S \subseteq G_i$. If each G_i is amenable, so is G.*

Corollary 19.2. *If every finitely generated subgroup of G is amenable, so is G.*

Proof of 19.1. For each $A \subseteq G$ and $r \in \mathbb{R}$, introduce a propositional variable $p_{A,r}$. Intuitively, $p_{A,r}$ is true iff $\mu(A) \geq r$. Let Φ be the following set of formulas:

1. $p_{A,0}$ for each $A \subseteq G$.
2. $p_{A,q} \Rightarrow p_{A,r}$ for $A \subseteq G$, $q \geq r$.
3. $(p_{A,q} \wedge p_{B,r}) \Rightarrow p_{A \cup B, q+r}$ for $A, B \subseteq G$, $A \cap B = \emptyset$, $q, r \in \mathbb{R}$
4. $(\neg p_{A,q} \wedge \neg p_{B,r}) \Rightarrow \neg p_{A \cup B, q+r}$ for $A, B \subseteq G$, $A \cap B = \emptyset$, $q, r \in \mathbb{R}$
5. $p_{A,q} \Rightarrow p_{B,q}$ for $A \subseteq B \subseteq G$, $q \in \mathbb{R}$
6. $p_{G,r}$ for each $r \leq 1$
7. $\neg p_{G,r}$ for each $r > 1$
8. $p_{A,r} \Leftrightarrow p_{gA,r}$ for $A \subseteq G$, $g \in G$, $r \in \mathbb{R}$

If a valuation v satisfies Φ, then define $\mu : \mathcal{P}(G) \to [0,1]$ by

$$\mu(A) = \sup \{r \in \mathbb{R} : v(p_{A,r}) = 1\}$$

This works just as in the proof of the fam extension theorem. To find such a v we'll use the Compactness Theorem. Fix finite $\Phi_0 \subseteq \Phi$. Let $\{g_i\}_{i \leq n} \subseteq G$ be all of the group elements appearing in Φ_0. In fact, Φ_0 need not be finite as long as only finitely many group elements appear. By assumption, there is an amenable $G_i \leq G$ with $\{g_i\}_{i \leq n} \subseteq G_i$. Let μ_i be a measure witnessing the amenability of G_i. Define $\tilde{\mu} : \mathcal{P}(G) \to [0,1]$ by

$$\tilde{\mu}(A) = \mu_i(A \cap G_i)$$

Then $\tilde{\mu}$ is a fam on $\mathcal{P}(G)$ with $\tilde{\mu}(G) = 1$. It's also left G_i-invariant:

$$\begin{aligned}
\tilde{\mu}(gA) &= \mu_i(gA \cap G_i) \\
\tilde{\mu}(gA) &= \mu_i(gA \cap gG_i) \\
\tilde{\mu}(gA) &= \mu_i(g(A \cap G_i)) \\
\tilde{\mu}(gA) &= \mu_i(A \cap G_i) \\
\tilde{\mu}(gA) &= \tilde{\mu}(A)
\end{aligned}$$

Therefore, Φ_0 is satisfiable and by compactness Φ is. □

Definition 19.3. A group satisfies the *Følner condition* if for each finite $S \subseteq G$, $\varepsilon > 0$, there is a finite nonempty $F \subseteq G$ so that for all $g \in S$, $\frac{|gF \triangle F|}{|F|} < \varepsilon$.

Example 19.4. If G is finite, let $F = G$.

Example 19.5. $(\mathbb{Z}, +)$. Fix finite $S \subseteq \mathbb{Z}$. Find k so that $S \subseteq [-k, k]$. Take n so that $n > \frac{k}{\varepsilon}$ and let $F = [-n, n]$. Then if $m \in F$,

$$\frac{|(m+F) \triangle F|}{|F|} \leq \frac{2\,|m|}{2n+1} \leq \frac{2k}{2n+1} < \varepsilon.$$

Example 19.6. $(\mathbb{Z}^2, +)$. Let $S \subseteq [-k, k]^2, \varepsilon > 0$. Let $F = [-n, n]^2$. Then if $g \in S$,

$$\frac{|(F+g) \triangle F|}{|F|} \leq \frac{4(2n+1)k}{(2n+1)^2} = \frac{4k}{2n+1}.$$

So we can clearly pick n large enough so that the quotient is less than ε.

Proposition 19.7. *A countably infinite group G satisfies the Følner condition iff there is a sequence (F_n) of nonempty finite subsets of G such that*

$$\forall g \in G \left[\lim_{n\to\infty} \frac{|gF_n \triangle F_n|}{|F_n|} = 0\right]$$

(F_n) is called a Følner sequence.

Proof. Let $G = \{g_1, g_2, \ldots\}$ satisfy the Følner condition. For each $n > 0$, let F_n be finite nonempty such that for all $g \in \{g_1, \ldots, g_n\}$, $\frac{|gF_n \triangle F_n|}{|F_n|} < \frac{1}{n}$. Then (F_n) is a Følner sequence.

Fix a Følner sequence (F_n). Let $S \subseteq G$ be finite, $\varepsilon > 0$. let $N \in \mathbb{Z}^+$ be large enough that

$$\forall n \geq N \forall g \in S \left[\frac{|gF_n \triangle F_n|}{|F_n|} < \varepsilon\right]$$

Take $F = F_N$; G satisfies the Følner condition. □

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20 Amenable Groups (cont.)

Theorem 20.1. *Let G be a group. The following are equivalent:*

1. *G is amenable*
2. *G satisfies the Følner condition*
3. *G is not paradoxical.*

Before we can prove this, we'll need a theorem of Hall.

Definition 20.2. If V is a set and $E \subseteq V^2$ is symmetric and irreflexive, then (V, E) is a *graph*. A *bipartite graph* has a vertex partition $V = X \sqcup Y$ with $\neg xEx'$ if $x, x \in X$ and $\neg yEy'$ if $y, y' \in Y$. A *perfect* $(1,k)$*-matching* $M \subseteq E$ is such that for all $x \in X$, there are exactly k distinct $y \in Y$ so that xEy and for all $y \in Y$, there is exactly one $x \in X$ with xEy. Equivalently, a perfect $(1,k)$-matching is a k-to-1 function $\varphi : Y \to X$ (that is, $\left|\varphi^{-1}(x)\right| = k$) with $\operatorname{graph}(\varphi) \subseteq E$. A perfect $(1,1)$-matching is often called a perfect matching. If $A \subseteq V$, then the *neighbors of A* are $N(A) = \{v \in V : \exists a \in A[aEv]\}$.

Theorem 20.3 (Hall $(1,k)$-matching theorem)**.** *Let F be a locally finite bipartite graph (every vertex has only finitely many neighbors) with vertices $X \sqcup Y$. There is a perfect $(1,k)$-matching iff for all finite $A \subseteq X, B \subseteq Y$, $|N(A)| \geq k\,|A|$ and $|N(B)| \geq \frac{1}{k}\,|B|$.*

Proof of 20.1. We already have $1 \Rightarrow 3$.

(2⇒1) Assume G is countable. Then fix a Følner sequence (F_n). Let μ be a fam on $\mathcal{P}(\mathbb{N})$ such that $\mu(\{n\}) = 0$, $\mu(\mathbb{N}) = 1$. Define $\nu : \mathcal{P}(G) \to [0,1]$ by

$$\nu(A) = \int_{\mathbb{N}} \frac{|A \cap F_n|}{|F_n|} \mathrm{d}\mu(n)$$

Clearly ν is a fam on F and $\nu(G) = 1$. Fix $g \in G$.

$$\begin{aligned}
|\nu(gA) - \nu(A)| &= \left| \int_{\mathbb{N}} \left(\frac{|gA \cap F_n|}{|F_n|} - \frac{|A \cap F_n|}{|F_n|} \right) \mathrm{d}\mu(n) \right| \\
|\nu(gA) - \nu(A)| &\leq \int_{\mathbb{N}} \left| \frac{|gA \cap F_n|}{|F_n|} - \frac{|A \cap F_n|}{|F_n|} \right| \mathrm{d}\mu(n) \\
|\nu(gA) - \nu(A)| &\leq \int_{\mathbb{N}} \left| \frac{\left|A \cap g^{-1} F_n\right|}{|F_n|} - \frac{|A \cap F_n|}{|F_n|} \right| \mathrm{d}\mu(n) \\
|\nu(gA) - \nu(A)| &\leq \int_{\mathbb{N}} \frac{\left|g^{-1} F_n \triangle F_n\right|}{|F_n|} \mathrm{d}\mu(n) \\
|\nu(gA) - \nu(A)| &= 0 \\
\nu(gA) &= \nu(A)
\end{aligned}$$

Now let G be arbitrary. For each finite $S \subseteq G$, $\varepsilon \in \mathbb{Q}^+$, let $F(S, \varepsilon) \subseteq G$ be finite nonempty with

$$\forall g \in S \left[\frac{|gF(S,\varepsilon) \triangle F(S,\varepsilon)|}{|F(S,\varepsilon)|} < \varepsilon \right]$$

For each finite $S \subseteq G$, let $S_0 = \langle S \rangle$ and

$$S_{n+1} = S_n \cup \bigcup_{\substack{\text{finite } S' \subseteq S \\ \varepsilon \in \mathbb{Q}^+}} F(S', \varepsilon)$$

$S_0 \leq S_1 \leq \cdots$ with each S_n countable. Then $S_\infty = \bigcup_n S_n$ is countable and satisfies the Følner condition. Then S_∞ is amenable containing S. S was an arbitrary finite set, so by Theorem 19.1, G is amenable.

(3⇒2) Assume G fails the Følner condition — we'll show that G is paradoxical. Let $S_0 \subseteq G$ and $2\varepsilon_0 > 0$ be such that for every finite nonempty $F \subseteq G$, there is $s_0 \in S_0$, $\frac{|s_0 F \triangle F|}{|F|} \geq 2\varepsilon_0$. We'll need some lemmas:

Lemma 1 There is $\lambda > 1$ and a finite $S_\lambda \subseteq F$ with $1 \in S_\lambda$ and for all nonempty finite $F \subseteq G$, $|S_\lambda F| \geq \lambda |F|$.

Proof of lemma 1 We have $|s_0 F \triangle F| = 2 |F \setminus s_0 F|$. Let $S_\lambda = S_0 \cup \{1\}$. Then for each finite $F \subseteq G$, $S_\lambda F \supseteq F$ and $S_\lambda F \setminus F = S_0 F \setminus F$. So

$$|S_\lambda F| - |F| = |S_0 F \setminus F| \geq |s_0 F \setminus F| = |F \setminus s_0 F| \geq \varepsilon_0 |F|$$

Then $|S_\lambda F| \geq (1 + \varepsilon_0) |F|$. Set $\lambda = 1 + \varepsilon_0$.

Lemma 2 (Amplification) There is a nonempty finite $S \subseteq G$ so that for all finite nonempty $F \subseteq G$, $|SF| \geq 2 |F|$.

Proof of lemma 2 Pick n so that $\lambda^n \geq 2$. Then $|S_\lambda^n F| = \left|S_\lambda(S_\lambda^{n-1}F)\right| = \lambda\left|S_\lambda^{n-1}F\right|$. By induction, $|S_\lambda^n F| \geq 2|F|$.

(3⇒2) Consider the graph $(X \sqcup Y, E)$ where $X = G = Y$ and $gEh \Leftrightarrow \exists s \in S[sg = h]$. If $A \subseteq X$ is finite, then $N(A) = |SA| \geq 2|A|$. Then take finite $B \subseteq Y$. $N(B) = S^{-1}B \supseteq s_0^{-1}B$, so $|N(B)| \geq \left|s_0^{-1}B\right| = |B| \geq \frac{1}{2}|B|$. By 20.3, there exists a perfect $(1,2)$-matching. In other words, there is a 2-to-1 surjection $\varphi : G \to G$. For $h \in G$ let $\varphi^{-1}(h) = \{\psi_1(h), \psi_2(h)\}$ (here we invoke Choice). Now $G = \psi_1(G) \sqcup \psi_2(G)$. For $s \in S$, let $A_s = \{h : \psi_1(h) = sh\}$. Similarly, let $B_s = \{h : \psi_2(h) = sh\}$. Then $G = \bigsqcup_{s\in S} A_s = \bigsqcup_{s\in S} B_s$ and $\psi_1(G) = \bigsqcup_{s\in S} sA_s$, $\psi_2(G) = \bigsqcup_{s\in S} sB_s$. Therefore, G is paradoxical. □

Corollary 20.4. *If G is exponentially bounded, then G is amenable.*

Proof of 20.3. The forward direction is trivial.

For the backwards direction, first assume the $k = 1$ case; we'll prove the general case from it. Let $G = (X \sqcup Y, E)$ be a graph satisfying the hypothesis of the theorem (we'll refer to this as *Hall's condition* from now on). Define $G' = (\bigsqcup_i^k X_i \sqcup Y, E')$ with X_i as copies of X. If $x \in X$, let x_i be its copy in X_i. We say $x_i E' y$ in G' if xEy in G.

Fix finite $A' \subseteq X'$. Define a projection $\overline{A'} = \{x \in X : \exists i[x_i \in A']\}$. Then $|A'| \leq k\left|\overline{A'}\right|$ and $N^{G'}(A') = N^G(\overline{A'})$ so

$$\left|N^{G'}(A')\right| = \left|N^G(\overline{A'})\right| \geq k\left|\overline{A'}\right| \geq |A'|$$

Now fix finite $B' \subseteq Y$. Then

$$\left|N^G(\overline{B'})\right| = k\left|N^G(B')\right| \geq k\frac{1}{k}|B'| = |B'|$$

Apply the $k = 1$ case to get a perfect matching M' on G'. Then define a perfect $(1,k)$-matching M on G by

$$(x, y) \in M \Leftrightarrow \exists i(x_i, y) \in M'$$

The $k = 1$ Case Using a Schroeder–Bernstein argument, we can reduce this to a one-sided version:

If for all finite $A \subseteq X$ we have $|N(A)| \geq |A|$, then there exists an injection $\varphi : X \to Y$ such that $(x, \varphi(x))$ is an edge.

Finite case We'll induct on $|X| = m$. The $m = 1$ case is obvious.

Assume that there is $X' \subseteq X$ with $1 \leq |X'| < m$ and $|N(X')| = |X'|$. Use the induction hypothesis to find an injection $\varphi' : X' \to N(X')$ consisting of edges. Let $X'' = X \setminus X'$. Restrict the graph to $X'', Y \setminus N(X')$. If $B \subseteq X''$, apply Hall's condition to $B \cup X'$. Then Hall's condition is satisfied for this graph, so there is an injection $\varphi'' : X'' \to Y$ consisting of edges. Take $\varphi = \varphi' \sqcup \varphi''$.

Now assume that for all $X' \subseteq X$, $1 \leq |X'| < m$, we have $|N(X')| > |X'|$. Now let $A' = X \setminus \{x_0\}$ and $B' = Y \setminus \{y_0\}$ with $x_0 E y_0$. Then Hall's condition holds on the induced subgraph on $A' \sqcup B'$, so there is an injection $\varphi' : A' \to B'$ consisting of edges. Then $\varphi = \varphi' \sqcup (x_0 \mapsto y_0)$ is as desired.

X **is infinite** If Hall's condition holds for a graph, it hold in every connected component. Then it suffices to consider connected graphs. Then X, Y are countable, since the graph is locally finite. Let $X = \{x_1, x_2, \dots\}$ with $x_i \neq x_j$ when $i \neq j$. By the finite case, for each n, there is an injection $\varphi_n : \{x_1, \dots, x_n\} \to N(\{x_1, \dots, x_n\})$ consisting of edges. Consider the sequence $\varphi_n(x_1)$. Since the graph is locally finite, there is a constant subsequence $n_0 < n_1 < \cdots$, $\varphi_{n_k}(x_1) = y_1$. By the same argument, we can find a y_i for each x_i, and $x_i \mapsto y_i$ is the desired matching.

$\square$

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21 Amenable Groups (cont.)

Conjecture 21.1 (von Neumman Conjecture (M. Day)). *A group G is amenable iff $\mathbb{F}_2 \not\leq G$.*

In the 1980s, Olshaanski showed that this is false. In fact, there are nonamenable groups without any elements of infinite order.

Fact 21.2. *If $G \leq GL_n(\mathbb{R})$, then von Neumman's Conjecture holds.*

Proposition 21.3. *If G is amenable and G acts on X, then X is not paradoxical. In fact, there is a G-invariant fam $\mu : \mathcal{P}(X) \to [0, 1]$ with $\mu(X) = 1$.*

Proof. Let ν witness the amenability of G. Let $x_0 \in X$ and consider the map $f : G \to X$ given by $f(g) = g \cdot x_0$. Let $\mu = f_*\nu$, that is, $\mu(A) = \nu(f^{-1}(A))$. □

Definition 21.4. Let G act on X. This action is *free* if for every $g \neq 1$, $g \cdot x \neq x$.

Theorem 21.5. *Let G be a nonamenable group. If G acts freely on X, then X is G-paradoxical.*

Corollary 21.6. *If G acts freely on X, then G is amenable iff X is not G-paradoxical.*

Proof of Theorem 21.5. Let $A, B \subseteq G$ with $A \sqcup B = G$ and $G \sim A \sim B$. By Choice, let $C \subseteq X$ have one representative from every G-orbit of X. Now for $x \in C$, $A \cdot x \sqcup B \cdot x = G \cdot x$. Let $A^* = \bigcup_{x \in C} A \cdot x$ and $B^* = \bigcup_{x \in C} B \cdot x$. Then $X = A^* \sqcup B^*$. Let $A = \bigsqcup_{i=1}^n A_i$, $G = \bigsqcup_{i=1}^n H_i$ and $A_i = g_i H_i$. Let $A_i^* = \bigcup_{x \in C} A_i \cdot x$ and $H_i^* = \bigcup_{x \in C} H_i \cdot x$. Then $A^* = \bigsqcup_{i=1}^n A_i^*$, $X = \bigsqcup_{i=1}^n H_i^*$, and

$$g_i \cdot H_i^* = g_i \bigcup_{x \in C} H_i \cdot x = \bigcup_{x \in C} g_i H_i \cdot x = \bigsqcup_{i=1}^n A_i \cdot x = A_i^*$$

Then $X \sim_G A^*$ and similarly $X \sim_G B^*$. □

In fact, we can strengthen this theorem.

Definition 21.7. For each $x \in X$ is stabilizer G_x is defined by $G_x = \{g \in G : g \cdot x = x\} \leq G$. The action is free iff $G_x = \{1\}$ for each x.

Theorem 21.8. *Let G act on X. If for each $x \in X$, the stabilizer G_x is amenable, then G is amenable iff X admits a G-invariant fam $\mu : \mathcal{P}(X) \to [0,1]$ with $\mu(X) = 1$.*

Proof. As before, let C have one point from each orbit. For $x \in C$, look at the map $\pi_x(g) = g \cdot x$. If $y = g \cdot x$, then $\pi_x^{-1}(y) = gG_x$. So there is a one-to-one correspondence between $G \cdot x$ and the set of left cosets $\{gG_x\}$. Since each G_x is amenable, we can choose a G_x-invariant fam μ_x. By translation, we can define a fam μ_C for each left coset CG_x.

For each $A \subseteq G$, define $f_A : X \to [0,1]$ by

$$f_A(y) = \mu_{\pi_x^{-1}(y)}(A \cap \pi_x^{-1}(y))$$

where x is the point of C in the orbit of y.

Now define $\nu : \mathcal{P}(G) \to [0,1]$ by $\nu(A) = \int_X f_A \mathrm{d}\mu$. Clearly ν is a fam with $\nu(G) = 1$.

$$\begin{aligned}
f_{gA}(y) &= \mu_{\pi_x^{-1}(y)}(gA \cap \pi_x^{-1}(y)) \\
f_{gA}(y) &= \mu_{g^{-1}\pi_x^{-1}(y)}(A \cap g^{-1}\pi_x^{-1}(y)) \\
f_{gA}(y) &= \mu_{\pi_x^{-1}(g^{-1}\cdot y)}(A \cap \pi_x^{-1}(g^{-1} \cdot y)) \\
f_{gA}(y) &= f_A(g^{-1} \cdot y)
\end{aligned}$$

Then

$$\begin{aligned}
f_{gA} &= (f_A)_{g^{-1}} \\
\nu(gA) &= \int f_{gA} \mathrm{d}\mu \\
\nu(gA) &= \int (f_A)_{g^{-1}} \mathrm{d}\mu \\
\nu(gA) &= \int f_A \mathrm{d}\mu \\
\nu(gA) &= \nu(A)
\end{aligned}$$

□

Proposition 21.9. $G_n \leq GL_{n+1}(\mathbb{R})$

Proof. Let $S \in G_n$. Write $S(\vec{x}) = U_S(\vec{x}) + \vec{a}_S$, where $U_S \in O_n$ and $\vec{a}_S \in \mathbb{R}^n$. In particular, $S \circ T(\vec{x}) = U_S \circ U_T(\vec{x}) + U_S(\vec{a}_T) + \vec{a}_S$. View U_S as an $n \times n$ matrix and write

$$M(S) = \begin{pmatrix} U_S & \vec{a}_S \\ \vec{0} & 1 \end{pmatrix} \in GL_{n+1}(\mathbb{R})$$

It's easy to check that $M : G_n \to GL_{n+1}(\mathbb{R})$ is an injective homomorphism. □

Proposition 21.10. *Let $G \subseteq SO_3$. G acts on S^2 by evaluation. The following are equivalent:*

1. *G is amenable,*
2. *$\mathbb{F}_2 \not\leq G$,*
3. *S^2 is not G-paradoxical,*
4. *There is a G-invariant fam on $\mathcal{P}(S^2)$.*

Proof. 1 and 2 are equivalent by 21.2. 3 and 4 are equivalent by Tarski's theorem. For the equivalence of 1 and 4, we apply theorem 21.8. For any x, every element of G_x is a rotation about the axis between $\vec{0}$ and $\vec{x}$. So G_x is abelian therefore amenable. □

22 The Banach–Tarski Paradox in $\mathbb{R}^n, n \geq 3$

Theorem 22.1. $\mathbb{F}_2 \leq SO_3$.

Proof. Let e_1, e_2, e_3 be the standard coordinate frame on $\mathbb{R}^3$. Let $0 < \theta < \frac{\pi}{2}$ with $\cos\theta = \frac{1}{3}$. Let R rotate around e_3 by θ and S rotate around e_1 by θ. We'll show that $\langle R, S\rangle \cong \mathbb{F}_2$. We'll show that any reduced word in $\left\{R^{\pm 1}, S^{\pm 1}\right\}$ ending in $R^{\pm 1}$ is not the identity. (Not that if v is a reduced word ending in $S^{\pm 1}$, v is the identity iff RvR^{-1} is.)

Let w be a reduced word ending in $R^{\pm 1}$. We claim that $w(e_1) = \left(a3^{-k}, b\sqrt{2}3^{-k}, c3^{-k}\right)$ with $a, b, c \in \mathbb{Z}$ and $3 \nmid b$. (In particular, $b \neq 0$ so $w(e_1) \neq e_1$)

Proof by induction on $|w|$. First, let $|w| = 1$, so $w = R^{\pm 1}$.

$$R^{\pm 1} e_1 = \begin{pmatrix} \cos\theta & \mp\sin\theta & 0 \\ \pm\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} (1, 0, 0) = \left(\frac{1}{3}, \pm\frac{2\sqrt{2}}{3}, 0\right)$$

Now let $|w| = n+1$ where the claim holds for words of length $\leq n$. If $w = R^{\pm 1} w'$, where $w' e_1 = 3^{-k}(a', b'\sqrt{2}, c')$, then

$$w(e_1) = R^{\pm 1} 3^{-k}(a', b'\sqrt{2}, c') = 3^{-k-1}(a, b\sqrt{2}, c)$$

where $a = a' \mp 4b', b = \pm 2a' + b', c = 3c'$. Similarly, if $w = S^{\pm 1} w'$, then

$$w(e_1) = S^{\pm 1} 3^{-k}(a', b'\sqrt{2}, c') = 3^{-k-1}(a, b\sqrt{2}, c)$$

where $a = 3a', b = b' \mp 2c', c = c' \pm 4b$.

This proves the claim that $a, b, c \in \mathbb{Z}$; all that's left to show is $3 \nmid b$. We check the $n = 2$ case by hand. We have $a', b', c' = (1, \pm 2, 0)$. Then either $w = R^{\pm 1} R^{\pm 1}$ and $b = \pm 2a' + b' = \pm 4$ or $w = S^{\pm 1} R^{\pm 1}$ and $b = b' \mp 2c' = 2$.

Now let $|w| > 2$ and assume $3 \nmid b$ holds for words of length $n - 1$.

1. $w = R^{\pm 1} S^{\pm 1} v$.

   $b = \pm 2a' + b'$, where $a' = 3a''$, so $3 \nmid b$.

2. $w = S^{\pm 1} R^{\pm 1} v$

   Similarly, $b = b' \mp 2c'$ where $c' = 3c''$, so $3 \nmid b$.

3. $w = R^{\pm 1} R^{\pm 1} v$

   $b = \pm 2a' + b'$ where $a' = a'' \mp 4b'$ and $b' = \pm 2a'' + b''$. So $b = 2b' - 9b''$ and $3 \nmid b$.

4. $w = S^{\pm 1} S^{\pm 1} v$

   $b = b' \mp 2c'$ where $b' = b'' \mp 2c''$ and $c' = c'' \pm 4b''$ so $b = 3b' - 9b''$ and $3 \nmid b$.

$\square$

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23 The Banach–Tarski Theorem in dimension $\geq$ 3 (cont.)

Let $G = \langle R, S \rangle \cong \mathbb{F}_2$ be the subgroup of SO_3 discussed last time. Let $D = \{x \in S^2 : \exists g \in G[g \neq 1 \wedge g \cdot x = x]\}$. D is countable and fixed by G:

Lemma 23.1. *D is G-invariant.*

Proof. Let $x \in D$ so that for some $h \in G$, $h \neq 1$, $h \cdot x = x$. Then for $g \in G$, we have $(gh^{-1}g) \cdot (g \cdot x) = gh \cdot x = g \cdot (h \cdot x) = g \cdot x$. Since $gh^{-1}g \in G$, we have $g \cdot x \in D$. □

Then $S^2 \setminus D$ is also G-invariant.

Theorem 23.2 (Hausdorff)**.** *There is a countable set $D \subseteq S^2$ such that $S^2 \setminus D$ is SO_3-paradoxical (and therefore G_3-paradoxical)*

Proof. Consider G, D as before. G acts freely on $S^2 \setminus D$, so we're done. □

Proposition 23.3. *For any countable set $D \subseteq S^2$, $S^2 \setminus D \sim_{SO_3} S^2$.*

Proof. We claim that there is a rotation so that $D, R(D), R^2(D), \ldots$ are pairwise disjoint. Assuming that, put $A = \bigsqcup_{n=0}^{\infty} R^n(D)$ and $B = S^2 \setminus A$. Then $S^2 = A \sqcup B$ and $S^2 \setminus D = R(A) \sqcup B$, so $S^2 \sim_{SO_3} S^2 \setminus D$.

Since D is countable, there is a line through the origin avoiding D. Let R_θ be a rotation about this line by an angle θ. Let $X = \{0 < \theta < 2\pi : \exists x \in D \exists y \in D \exists n \in \mathbb{N}[R_\theta^n(x) = y]\}$. For every $x, y \in D$, $n \in \mathbb{N}$, there are finitely many θ such that $R_\theta^n(x) = y$. Therefore X is countable. In particular, we can pick $\theta \notin X$. Then for each $n \in N$, $R_\theta^n(D) \cap D = \emptyset$. Then for $m > n$, we have $R_\theta^m(D) \cap R_\theta^n(D) = R_\theta^n(R_\theta^{m-n}(D) \cap D) = \emptyset$ □

Theorem 23.4 (Banach–Tarski, I)**.** *S^2 is SO_3-paradoxical.*

Theorem 23.5 (Banach–Tarski, II)**.** *A closed ball in $\mathbb{R}^3$ is paradoxical.*

Proof. It's enough to consider a ball centered at 0. WLOG, assume the ball has radius 1. Using the shrinking map $\vec{x} \to \alpha\vec{x}$, $0 < \alpha \leq 1$, it is clear that $B \setminus \left\{\vec{0}\right\}$ is SO_3-paradoxical. Now consider a circle containing the origin which is contained in the unit ball. Rotating this circle by $\alpha\pi$ with α irrational shows that $B \setminus \left\{\vec{0}\right\} \sim B$. □

Theorem 23.6 (Banach–Tarski, III)**.** *If $A, B \subseteq \mathbb{R}^3$ are bounded and have nonempty interor, then $A \sim B$. In particular, any such set is paradoxical.*

Proof. By Banach–Schroeder–Bernstein, it is enough to show that $A \prec B$. Choose closed balls K, L with $A \subseteq K$, $L \subseteq B$. Then we just need to show that $K \prec L$. By 23.5, we know that $L \sim \bigsqcup_{i=1}^n L_i$, where the L_i are pairwise disjoint balls of the same radius as L. For large enough n, we can find $L_1', \ldots, L_n'$ of the same radius as L such that $K \subseteq \bigcup_{i=1}^n L_i'$. Let $L_1'' = L_1'$ and $L_i'' = L_i' \setminus \bigcup_{j<i} L_j''$. Then $K \subseteq \bigsqcup_{i=1}^n L_i'' \prec \bigsqcup_{i=1}^n L_i \sim L$ and $K \prec L$. □

Corollary 23.7. *There is no isometry-invariant fam $\mu : \mathcal{P}(\mathbb{R}^3) \to [0, \infty]$ such that $0 < \mu(B) < \infty$ where B is a unit ball. In particular, there is no such extension of Lebesgue measure.*

Theorem 23.8 (Banach–Tarski for dimension ≥ 3)**.** *Let $n \geq 3$. Then*

1. *S^{n-1} is SO_n-paradoxical.*
2. *Any ball in $\mathbb{R}^n$ is paradoxical.*
3. *Any two bounded sets in $\mathbb{R}^n$ with nonempty interior are equidecomposable.*
4. *There is no isometry-invariant fam on $\mathcal{P}(\mathbb{R}^n)$ such that $0 < \mu(B) < \infty$ for B a unit ball. In particular, there is no such extension of Lebesgue measure.*

Proof. It's enough to prove 1 by induction on n. We have the $n = 3$ case already.

By the inductive hypothesis, S^{n-1} is SO_n-paradoxical. Let $S^{n-1} \sim_{SO_n} A \sim_{SO_n} B$, say $S^{n-1} = \bigsqcup_{i=1}^m K_i$ and $A = \bigsqcup_{i=1}^m L_i$, $R_i \in SO_n$, $R_i(K_i) = L_i$, $B = \bigsqcup_{i=1}^m M_i$, $S_i \in SO_n$, $S_i(M_i) = L_i$.
Define $\varphi : (S^n)^* = S^n \setminus \{(0, \ldots, 0, \pm 1)\} \to S^{n-1}$ by

$$\varphi(x_1, \ldots, x_n, x_{n+1}) = \frac{(x_1, \ldots, x_n)}{|(x_1, \ldots, x_n)|}.$$

Define $R_i^*, S_i^* \in SO_{n+1}$ by

$$R_i^* = \begin{pmatrix} R_i & 0 \\ 0 & 1 \end{pmatrix} \quad S_i^* = \begin{pmatrix} S_i & 0 \\ 0 & 1 \end{pmatrix}.$$

Let $K_i^* = \varphi^{-1}(K_i)$ and define L_i^*, M_i^*, N_i^* similarly. Then $(S^n)^* = \bigsqcup_{i=1}^m K_i^*$ and $A^* = \varphi^{-1}(A) = \bigsqcup_{i=1}^m L_i^*$ with $R_i^*(K_i^*) = L_i^*$. Similarly, $B^* = \varphi^{-1}(B) =$

$\bigsqcup_{i=1}^{m} M_i^*$ with $S_i^*(K_i^*) = M_i^*$. So $(S^n)^* = A^* \sqcup B^*$ and $(S^n)^* \sim_{SO_{n+1}} A^* \sim_{SO_{n+1}} B^*$. Therefore, $(S^n)^*$ is SO_{n+1} paradoxical.

Next, we'll show that $S^n \sim (S^n)^*$. Let $D = \{(0, \ldots, 0, \pm 1)\}$. Define $T \in SO_{n+1}$ so that $T^m(D) \cap T^n(D) = \emptyset$ for $m \neq n$. Say $T(x_1, \ldots, x_{n-1}, x, y) = (x_1, \ldots, x_{n-1}, (x+iy)e^{i\pi\theta})$ with θ irrational. Then $S^n \sim S^n \setminus D$. $\square$

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24 Tarski's Theorem

Theorem 24.1. *Let G act on X and $A \subseteq X$. Then A is not G-paradoxical iff there exists a G-invariant fam $\mu : \mathcal{P}(X) \to [0, \infty]$ such that $\mu(A) = 1$.*

We've already given the backward direction. For the forward direction, we'll need new machinery.

Definition 24.2. If G acts on X, define $X^* = X \times \mathbb{N}$ and $G^* = G \times S_\infty$, where S_∞ is the group of permutations of $\mathbb{N}$. Then G^* acts on X^* by $(g, \sigma) \cdot (x, n) = (g \cdot x, \sigma(n))$.

Definition 24.3. A set $A \subseteq X^*$ is *bounded* if for some n,

$$A \subseteq \bigcup_{m \leq n} X \times \{m\}$$

Define B as the collection of all bounded subsets of X^*. Let $\mathcal{S} = B/\sim_{G^*}$ be the set of all $[A] = [A]_{\sim_{G^*}}$ for $A \in B$.

Proposition 24.4.

1. *For $A, B \subseteq X$, $A \sim_G B$ iff $A \times \{m\} \sim_{G^*} B \times \{n\}$ for $m, n \in \mathbb{N}$.*
2. *A is G-paradoxical iff $A \times \{0\} \sim_{G^*} A \times \{0, 1\}$.*

Proof.

1 Let σ be such that $\sigma(m) = n$ and let $(g_i)_{i \leq k}$ witness the equidecomposability of A and B. Then $((g_i, \sigma))_{i \leq k}$ witnesses the equidecomposability of $A \times \{m\}$ and $B \times \{n\}$. For the reverse direction, just project $(g_i, \sigma_i) \mapsto g_i$.

2 Assume A is G-paradoxical. Let $A = B \sqcup C$ with $A \sim_{G^*} B \sim_{G^*} C$. Then $B \times \{0\} \sim_{G^*} A \times \{0\}$ and $C \times \{0\} \sim_{G^*} A \times \{1\}$ by the above.

Now assume that $A \times \{0\} \sim_{G^*} A \times \{0, 1\}$. Let $A \times \{0\} = \bigsqcup_{i=1}^k A_i$ and $A \times \{0,1\} = \bigsqcup_{i=1}^k A'_i$ with $(g_i, \sigma_i)A_i = A'_i$. Separate the A'_i according to whether $\sigma_i(0) = 0$ or $\sigma_i(0) = 1$, so that $A \times \{0\} = \bigsqcup_{i=1}^m C_i \sqcup \bigsqcup_{i=1}^n D_i$ and $A \times \{0\} = \bigsqcup_{i=1}^m C'_i$, $A \times \{1\} = \bigsqcup_{i=1}^n D'_i$ with $g_i \cdot C_i = C'_i$ and $h_i \cdot D_i = D'_i$. Put $C = \bigsqcup_{i=1}^m C_i$ and $D = \bigsqcup_{i=1}^n D_i$. We have $C \sim_{G^*} A \times \{0\} \sim_{G^*} A \times \{1\} \sim_{G^*} D$, so that $C \sim_G A \sim_G D$ and A is G-paradoxical. □

From now on, identify X with $X \times \{0\}$.

Definition 24.5. Given $[A], [B] \in \mathcal{S}$, define $[A] + [B] = [A' \cup B']$ where $A' \sim_{G^*} A, B' \sim_{G^*} B$ and $A' \cap B' = \emptyset$.

This is well-defined. Let A'', B'' satisfy the same conditions as A', B'. Then $A'' \sim_{G^*} A \sim_{G^*} A'$ and $B'' \sim_{G^*} B \sim_{G^*} B'$, so $A'' \cup B'' \sim_{G^*} A' \cup B'$.

Definition 24.6. Let $n[A] = [A] + (n-1)[A]$ and $1[A] = [A]$. By 24.4, A is paradoxical iff $[A] = 2[A]$. Let $0 = [\emptyset]$.

Proposition 24.7. $(\mathcal{S}, +, 0)$ *is an abelian semigroup with identity.*

Proof. Let $\alpha = [A], \beta = [B], \gamma = [C]$. Let $A' \sim_{G^*} A, B' \sim_{G^*} B, C' \sim_{G^*} C$ with A', B', C' pairwise disjoint. Then $(\alpha+\beta)+\gamma = [(A' \cup B') \cup C'] = [A' \cup (B' \cup C')] = \alpha + (\beta + \gamma)$. Similarly, $\alpha + \beta = [A' \cup B'] = [B' \cup A'] = \beta + \alpha$. Finally, $\alpha + 0 = [A' \cup \emptyset] = [A'] = \alpha$. □

Definition 24.8. Define a relation on $\mathcal{S}$ by $\alpha \leq \beta$ iff $\exists \gamma (\alpha + \gamma = \beta)$.

Proposition 24.9.

1. $\leq$ *is partial order on* $\mathcal{S}$ *with minimal element* 0.
2. $\alpha \leq \beta \Rightarrow \alpha + \gamma \leq \beta + \gamma$.
3. $[A] \leq [B] \Leftrightarrow A \prec_{G^*} B$.

Proof.

3. $A \prec_{G^*} B$ iff there is some C so that $A \sqcup C \sim_{G^*} B$. In other words, $[A] + [C] = [B]$.
2. $\alpha + \delta = \beta$. Then $\alpha + \gamma + \delta = \beta + \gamma$, so $\alpha + \gamma \leq \beta + \gamma$.
1. Reflexivity is trivial. For transitivity, assume $\alpha \leq \beta \leq \gamma$. Then $\beta = \alpha + \delta$ and $\gamma = \beta + \varepsilon = \alpha + \delta + \varepsilon$, so $\alpha \leq \gamma$. Antisymmetry follows from 3 and the Banach–Schroeder–Bernstein Theorem.

□

Theorem 24.10 (Cancellation Law)**.** *If* $\alpha, \beta \in \mathcal{S}, n > 0$*, then* $n\alpha = n\beta \Rightarrow \alpha = \beta$.

Corollary 24.11. *If $\alpha \in \mathcal{S}$ with $(n+1)\alpha \leq n\alpha$, then $\alpha = 2\alpha$.*

Proof.

$$n\alpha \geq (n+1)\alpha = n\alpha + \alpha \geq (n+1)\alpha + \alpha = n\alpha + 2\alpha \geq \cdots \geq n\alpha + n\alpha = 2n\alpha$$

But also $n\alpha \leq 2n\alpha$, so by cancellation, $\alpha = 2\alpha$. $\square$

In any abelian semigroup $(\mathcal{T}, +, 0)$ with identity, we can define a quasiorder $\alpha \leq \beta \Leftrightarrow \exists\gamma(\alpha + \gamma = \beta)$. (We don't necessarily have antisymmetry.)

Definition 24.12. A *fam on $\mathcal{T}$* is a map $\mu : \mathcal{T} \to [0, \infty]$ such that $\mu(0) = 0$ and $\mu(\alpha + \beta) = \mu(\alpha) + \mu(\beta)$.

Theorem 24.13. *Let $(\mathcal{T}, +, 0)$ be an abelian semigroup with identity and let $\alpha_0 \in \mathcal{T}$. Then the following are equivalent:*

- *$(n+1)\alpha_0 \nleq n\alpha_0$ for all $n \in \mathbb{N}$*
- *There is a fam μ on $\mathcal{T}$ with $\mu(\alpha_0) = 1$.*

Proof of Tarski's Theorem. Fix $A \subseteq X$ with A not G-paradoxical. Then $[A] \neq 2[A]$. By 24.11, $(n+1)\alpha \nleq n\alpha$ for all $n \in \mathbb{N}$. By 24.13 for $(\mathcal{S}, +, 0)$, there is a fam $\nu : \mathcal{S} \to [0, \infty]$ such that $\nu([A]) = 1$. Then define a fam $\mu : \mathcal{P}(X) \to [0, \infty]$ by $\mu(B) = \nu([B])$. Let's check the axioms:

- $\mu(\emptyset) = \nu(0) = 0$
- If $B \cap C = \emptyset$, then $\mu(B \cup C) = \nu([B \cup C]) = \nu([B] + [C]) = \nu([B]) + \nu([C]) = \mu(B) + \mu(C)$
- $\mu(gB) = \nu([gB]) = \nu([B]) = \mu(B)$.

$\square$

We'll need the following result to prove the Cancellation Law.

Theorem 24.14 (König)**.** *Every k-regular bipartite graph has a perfect matching. (A graph is k-regular if the degree of every vertex is k.)*

Proof. Call the parts of the graph A and B. By Hall's theorem, it suffices to check that for finite $X \subseteq A$ and $Y \subseteq B$, $|N(X)| \geq |X|$ and $|N(Y)| \geq |Y|$. Fix $X \subseteq A$. Let x be the number of edges coming out of X, so $x = k\,|X|$. But also $k\,|N(X)| \geq x$. Therefore, $|N(X)| \geq |X|$. $\square$

Proof of Cancellation Law. Say $[A_1] = \alpha, [B_1] = \beta$. Let $A = \bigsqcup_{i=1}^n A_i$ and $B = \bigsqcup_{i=1}^n B_i$. Then $n\alpha = [A]$ with $[A_i] = [A_1]$ and similarly $n\beta = [B]$ with $[B_i] = [B_1]$. Let φ witness the equidecomposability of A and B. Let f_i witness $A_1 \sim_{G^*} A_i$, g_i witness $B_1 \sim_{G^*} B_i$.

Define a bipartite graph on $A_1 \sqcup B_1$ by the edge relation aEb if $\exists i \exists j [b = g_j^{-1}(\varphi(f_i(a)))]$. This graph is n-regular, so by König's theorem, it has a perfect matching $\rho : A \to B$. Let $A_{ij} = \left\{a \in A_1 : \rho(a) = g_j^{-1}(\varphi(f_i(a)))\right\}$. Define B_{ij} similarly. Then $A_1 = \bigsqcup_{i,j} A_{ij}$ and $B_1 = \bigsqcup_{i,j} \rho(A_{ij}) = \bigsqcup_{i,j} B_{ij}$. Then $B_{ij} \sim_{G^*} A_{ij}$, giving us $A_1 \sim B_1$. $\square$

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25 Tarski's Theorem

We still have one unproven fact in our proof of Tarski's theorem, which we'll restate for convenience:

Theorem 25.1. *Let $(\mathcal{T}, +, 0)$ be an abelian semigroup with identity and let $\alpha_0 \in \mathcal{T}$. Then the following are equivalent:*

- $(n+1)\alpha_0 \not\leq n\alpha_0$ *for all* $n \in \mathbb{N}$
- *There is a fam* μ *on* $\mathcal{T}$ *with* $\mu(\alpha_0) = 1$.

Proof. For the forward direction, call $\alpha \in \mathcal{T}$ *small* if $\exists n[\alpha \leq n\alpha_0]$. Let $\mathcal{T}' = \{\alpha \in \mathcal{T} : \alpha \text{ is small}\}$. Then $\mathcal{T}'$ is a subsemigroup containing α_0. If there is a fam $\mu' : \mathcal{T}' \to [0, \infty]$ such that $\mu'(\alpha_0) = 1$, then there is a fam $\mu : \mathcal{T}' \to [0, \infty]$ such that $\mu(\alpha_0) = 1$, defined by

$$\mu(\alpha) = \begin{cases} \mu'(\alpha), & \text{if } \alpha \in \mathcal{T}' \\ \infty, & \text{otherwise} \end{cases}$$

So we can assume WLOG that $\mathcal{T}' = \mathcal{T}$. We need this lemma:

Lemma 25.2. *There is* $\mu : \mathcal{T} \to [0, \infty]$ *such that* $\mu(\alpha_0) = 1$ *and for* $\alpha_i, \beta_i \in \mathcal{T}$,

$$\sum_{i=1}^{n} \alpha_i \leq \sum_{j=1}^{m} \beta_j \Rightarrow \sum_{i=1}^{n} \mu(\alpha_i) \leq \sum_{j=1}^{m} \mu(\beta_j)$$

Note that such a μ is a fam:

- $0 + \alpha_0 \leq \alpha_0$, so $\mu(0) + \mu(\alpha_0) \leq \mu(\alpha_0)$ and $\mu(0) = 0$.
- $(\alpha + \beta) \leq \alpha + \beta$ and $\alpha + \beta \leq (\alpha + \beta)$, so $\mu(\alpha + \beta) \leq \mu(\alpha) + \mu(\beta) \leq \mu(\alpha + \beta)$.

By compactness, it's enough to prove the following finite version:

Claim If $\mathcal{T}_0 \subseteq \mathcal{T}$ is finite and $\alpha_0 \in \mathcal{T}_0$, then there exists $\mu_0 : \mathcal{T}_0 \to [0, \infty]$ such that $\mu_0(\alpha_0) = 1$ and for any $(\alpha_i), (\beta_i)$ in $\mathcal{T}_0$, we have $\sum_{i=1}^n \mu_0(\alpha_i) \leq \sum_{j=1}^m \mu_0(\beta_j)$. Note that these sums might lie outside of $\mathcal{T}_0$.

We'll prove this by induction on $|\mathcal{T}_0|$. If $\mathcal{T}_0 = \{\alpha_0\}$, then define $\mu_0(\alpha_0) = 1$. Our claim reduces to checking $m\alpha_0 \leq n\alpha_0 \Rightarrow m \leq n$. This follows from the assumption that $(n+1)\alpha_0 \not\leq n\alpha_0$.

Now let $\mathcal{T}_0 \neq \{\alpha_0\}$. Pick $\alpha \neq \alpha_0$ and apply the induction hypothesis to $\mathcal{T}_0 \setminus \{\alpha\}$ to get $\mu_0' : \mathcal{T}_0 \to [0, \infty]$. Notice that since all elements of $\mathcal{T}$ are small, μ_0' is actually finite-valued. To extend μ_0' to a μ_0, let

$$\mu_0(\alpha) = \inf \left\{ \frac{\sum_{i=1}^m \mu_0'(a_i) - \sum_{j=1}^n \mu_0'(b_j)}{r} \right\}$$

where $r \in \mathbb{Z}^+$, $a_i, b_j \in \mathcal{T}_0 \setminus \{\alpha\}$ and $\sum_{i=1}^n b_i + r\alpha \leq \sum_{j=1}^m a_j$. To show that this works, we need to show that if $\alpha_i, \beta_j \in \mathcal{T}_0 \setminus \{\alpha\}$, $s, t \in \mathbb{N}$ and $\sum_{i=1}^m \alpha_i + s\alpha \leq \sum_{j=1}^n \beta_j + t\alpha$, then $\sum_{i=1}^m \mu_0(\alpha_i) + s\mu_0(\alpha) \leq \sum_{j=1}^n \mu_0(\beta_j) + t\mu_0(\alpha)$. If $s = t = 0$, this follows directly from the induction hypothesis.

Case $s = 0$, $t > 0$ We need to show that $\mu_0(\alpha) \geq \frac{\sum \mu_0'(\alpha_i) - \sum \mu_0'(b_j)}{t}$. Equivalently,

$$\frac{\sum \mu_0'(a_k) - \sum \mu_0'(\beta_l)}{r} \geq \frac{\sum \mu_0'(\alpha_i) - \sum \mu_0'(\beta_j)}{t}$$

with $r \in \mathbb{Z}^+$, $a_k, b_l \in \mathcal{T}_0 \setminus \{\alpha\}$, and $\sum b_l + r\alpha \leq \sum a_k$, $\sum \alpha_i \leq \sum \beta_j + t\alpha$. Equivalently,

$$r \sum \mu_0'(\alpha_i) + t \sum \mu_0'(b_l) \leq r \sum \mu_0'(\beta_j) + t \sum \mu_0'(a_k)$$

By finite additivity, it's enough to check that $r \sum \alpha_i + t \sum b_l \leq r \sum \beta_j + t \sum a_k$. We have

$$\begin{aligned} r \sum \alpha_i &\leq r \sum \beta_j + rt\alpha, \\ t \sum b_l + rt\alpha &\leq t \sum a_k, \\ r \sum \alpha_i + t \sum b_l &\leq r \sum \beta_j + t \sum a_k. \end{aligned}$$

Case $s > 0$ Let $\sum b_l + r\alpha \leq \sum a_k$. Then we show that

$$\sum \mu_0'(\alpha_i) + s\mu_0'(\alpha) \leq \sum \mu_0'(\beta_j) + t \frac{\sum \mu_0'(a_k) - \sum \mu_0'(b_l)}{r}.$$

Equivalently,

$$\mu_0'(\alpha) \leq \frac{t \sum \mu_0'(a_k) + r \sum \mu_0'(\beta_j) - (t \sum \mu_0'(b_l) + r \sum \mu_0'(\alpha_i))}{rs}.$$

It's enough to check that

$$t\sum b_l + r\sum \alpha_i + rs\alpha \le t\sum a_k + r\sum \beta_j$$

This follows from $\sum b_l + r\alpha \le \sum a_k$ and $\sum \alpha_i + s\alpha \le \sum \beta_k + t\alpha$:

$$\begin{aligned} t\sum b_l + tr\alpha &\le t\sum a_k \\ r\sum \alpha_i + rs\alpha &\le r\sum \beta_j + tr\alpha \\ t\sum b_l + r\sum \alpha_i + rs\alpha &\le r\sum \beta_k + t\sum a_k \end{aligned}$$

□

26 Countable Equidecompositions

Consider $(X, \mathcal{A})$ with X a set and $\mathcal{A}$ a σ-subalgebra of subsets of X. Let G act on X such that for $A \in \mathcal{A}$ and $g \in G$, $g \cdot A \in \mathcal{A}$. Let's define a new notion of equidecomposability $A \sim_{G,\infty} B \Leftrightarrow A \sim_\infty B$ iff there exist $g_n \cdot A_n = B_n$ with $A = \bigsqcup_n^\infty A_n, B = \bigsqcup_n^\infty B_n$. We say X is *countably G-paradoxical* if $A = B \sqcup C$ and $A \sim_\infty B \sim_\infty C$. We'll consider countably additive measures now. We have a partial analogue of Tarski's Theorem:

Fact 26.1. *If there exists a G-invariant probability measure on $\mathcal{A}$, then X is not countably G-paradoxical.*

But in fact, the converse is not true:

Theorem 26.2 (Chuaqui)**.** *There is an X which is not countably G-paradoxical for which there is no G-invariant probability measure.*

Let ω_1 be the unique uncountable well-order such that if $\alpha \in \omega_1$, then $\{\beta : \beta \le \alpha\}$ is countable. It follows that any countable $A \subseteq \omega_1$ is bounded.

Theorem 26.3 (Ulam)**.** *There is no countably additive probability measure $\mu : \mathcal{P}(\omega_1) \to [0,1]$ with $\mu(\{\alpha\}) = 0$ for all $\alpha \in \omega_1$.*

Proof. Let $W_y = \{\alpha \in \omega_1 : \alpha < y\}$. Let $f^y : W_y \to \mathbb{N}$ be injective. Consider $f(x,y) = f^y(x)$ for $x < y$, $x, y \in \omega_1$. Then $x < x' < y \Rightarrow f(x,y) \ne f(x',y)$. For $n \in \mathbb{N}$, $x \in \omega_1$, put $F_x^n = \{y : x < y, f(x,y) = n\}$.

Consider the Ulam matrix with uncountably many columns and countably many rows, where the element in the n^{th} row and the α^{th} column is F_α^n. The sets in any row are pairwise disjoint. To see this, assume $F_x^n \cap F_{x'}^n \ne \emptyset$ with $x \ne x'$. Let $y \in F_x^n \cap F_{x'}^n$ with $x, x' < y$. Then $f(x,y) = n = f(x',y)$; contradiction.

Additionally, the union of a column is co-countable, i.e., $\bigcup_n F_x^n = \omega_1 \setminus C$ with C a countable set. To see this, let $x < y$ with $y \in F_x^n$, $f(x,y) = n$. So $\bigcup_n F_x^n \supseteq \omega_1 \setminus \{y : y \le x\}$.

Finally, let μ be a countable additive probability measure on $\mathcal{P}(\omega_1)$. If $(X_i)_{i\in F}$ is a family of subsets of ω_1, and $X_i \cap X_j = \emptyset$ if $i \neq j$, then $\{i \in F : \mu(X_i) > 0\}$ is countable because the set $\left\{i \in I : \mu(X_i) \geq \frac{1}{n}\right\}$ is finite. (In fact, it has at most n elements, since the measure of the whole space is 1).

So there is y_n such that $x > y_n \Rightarrow \mu(F_x^n) = 0$. Let $z > \sup_n y_n$ (again we use the boundedness of countable sets in ω_1). Then $\mu(F_z^n) = 0$, but $\bigcup_n F_z^n = \omega_1 \setminus C$. $\mu(C) = \sum_{x\in C} \mu(\{x\}) = 0$, so $\mu(\omega_1) = \mu(\omega_1 \setminus C) = \sum_n \mu(F_z^n) = 0$. Contradiction. $\square$

Proof of Chuaqui's Theorem. Take $X = \omega_1$. Let G be the group of permutations of ω_1 with finite support. (Recall that π has finite support if $\{\alpha \in \omega_1 : \pi\alpha \neq \alpha\}$ is finite) We claim that ω_1 is not countably G-paradoxical.

Assume towards a contradiction that $\omega_1 = A \sqcup B$ with $A \sim_\infty B \sim_\infty \omega_1$. Because $\omega_1 \sim_\infty A$, we have a bijection $g : \omega_1 \to A$ and a countable decomposition $\omega_1 = \bigsqcup_n^\infty A_n$ and $g_n \in G$ such that $g\restriction A_i = g_i \restriction A_i$. Then the support of g is at most the union of the supports of the g_i, so g has countable support. Similarly, there is a bijection $h : \omega_1 \to B$ with countable support. Let α be outside the supports of g and h. Then $g(\alpha) = \alpha = h(\alpha) \in A \cap B$; contradiction. Therefore, G is not countably G-paradoxical.

Assume $\mu : \mathcal{P}(\omega_1) \to [0,1]$ is a G-invariant probability measure. Then there must be some β for which $\mu(\{\beta\}) = 0$. For any α, there is $g_\alpha \in G$ so that $g_\alpha(\beta) = \alpha$. Then we must have $\mu(\{\alpha\}) = 0$ for all $\alpha \in \omega_1$, contradicting Ulam's theorem. $\square$

Definition 26.4. A group G is a *Polish group* if there is a metric $d : G \to [0,\infty)$ which is separable and complete and $g \mapsto gh, g \mapsto g^{-1}$ are continuous.

Example 26.5. $\langle \mathbb{R}^n, + \rangle, \langle G_n, \cdot \rangle$ are Polish groups.

Definition 26.6. X is a *Polish space* if it has a separable complete metric. Let $\mathcal{B}(X)$ be the Borel subsets of X.

A map $G \times X \to X$ is a *Borel action* if for every open set $U \subseteq X$, the set $\{(g,x) \in G \times X : g \cdot x \in U\}$ is Borel in $G \times X$.

Theorem 26.7 (Becker–Kechris)**.** *If G is a Polish group and X is a Polish space, $\mathcal{A} = \mathcal{B}(X)$ and G acts on X in a Borel way, the following are equivalent:*

- *X is not countably G-paradoxical with respect to $\mathcal{A}$*
- *There exists a G-invariant probability measure $\mu : \mathcal{A} \to [0,1]$.*

Proof sketch. We consider just the forward direction. First, reduce this to the case when $(g,x) \mapsto g \cdot x$ is continuous. Choose $H \leq G$ countable, dense. By continuity, it's enough to find a H-invariant measure. X is not countably G-paradoxical, so X is not countably H-paradoxical. Let E be the equivalence relation induced by this action. Then we apply

Theorem 26.8 (Nadkarni)**.** *If E is not compressible then there exists an H-invariant measure $\mu : \mathcal{B}(X) \to [0,1]$.*

E is *compressible* if there is a Borel $A \subseteq X$ and a Borel bijection $f : X \to A$ such that $f(x)Ex$ and $X \setminus A$ intersects every E-class. Note that $f(B) \cap B = \emptyset$.

Now assume E is compressible. Let $B = X \setminus A$. Let $B_n = f^n(B)$. If $n > m$, $f^n(B) \cap f^m(B) = \emptyset$. Each B_n meets every E-class and f^n is a bijection between B and B_n. let $H = \{h_n : n \in \mathbb{N}\}$ and let $N : X \to \mathbb{N}$ be such that $N(x)$ is the least n for which $h_n \cdot x \in B$. N is a Borel function. Let $p_0(x) = f^{2N(x)}(h_{N(x)} \cdot x)$ and $p_1(x) = f^{2N(x)+1}(h_{N(x)} \cdot x)$. Say $C = p_0(X)$ and $D = p_1(X)$. Clearly, $C \cap D = \emptyset$. Since f respects E, we can represent p_0 and p_1 as piecewise H-maps. Therefore, $X \sim_{H,\infty} C \sim_{H,\infty} D$; contradiction. $\square$

Problem 26.9. *Under the same hypotheses of Theorem 26.7, are the following equivalent for arbitrary $A \in \mathcal{A}$?*

- *A is not countably G-paradoxical with respect to $\mathcal{A}$*
- *There exists a G-invariant measure $\mu : \mathcal{A} \to [0, 1]$ with $\mu(A) = 1$.*

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27 Recent Results

27.1 Tarski Circle Squaring Problem

Recall that if $A, B \subseteq \mathbb{R}^2$ are bounded and Lebesgue measurable and $A \sim B$, then $m_2(A) = m_2(B)$.

Problem 27.1. *Let D, S be a closed disk and a square of the same area, respectively. Is $D \sim S$?*

Yes! Something stronger turns out to be true.

Theorem 27.2 (Laczkovich, 1990)**.** *Let $A, B \subseteq \mathbb{R}^n$ be bounded, Lebesgue measurable sets with $m_n(A) = m_n(B) > 0$ and boundaries of box dimension less than n. Then $A \sim_{\mathbb{R}^n} B$, i.e., they are equidecomposable using only translations.*

Theorem 27.3 (Grabowski–Mathé–Pikhurko, 2015)**.** *In the decomposition of the previous theorem, the pieces can be taken to be Lebesgue measurable.*

Problem 27.4. *Can this be done using Borel pieces?*

Theorem 27.5 (Grabowski–Mathé–Pikhurko, 2014)**.** *If $A, B \subseteq \mathbb{R}^n$, $n \geq 3$, are bounded, Lebesgue measurable with non-empty interior and $m_n(A) = m_n(B) > 0$, then $A \sim B$ with Lebesgue measurable pieces.*

27.2 Marczewski Problem

Problem 27.6 (Marczewski, 1930)**.** *Is the unit ball in $\mathbb{R}^3$ paradoxical using sets with the property of Baire?*

Theorem 27.7 (Dougherty–Foreman, 1994)**.** *If $A, B \subseteq \mathbb{R}^n$, $n \geq 3$, are bounded with nonempty interior and have the property of Baire, then $A \sim B$ using sets with the property of Baire.*

Definition 27.8. If $A \subseteq \mathbb{R}^n$, then A is *meager* if $A \subseteq \bigcup_n F_n$ where the F_n are closed with empty interior.

$A \subseteq \mathbb{R}^n$ has the *property of Baire* if there is a Borel set B (equivalently, an open set) such that $B \triangle A$ is meager. Notice that the class of sets with property of Baire forms a σ-algebra.

Theorem 27.9 (Marks–Unger, 2015)**.** *Let a group G act on a Polish space X by Borel automorphisms of X. Then X is G-paradoxical iff X is G-paradoxical using pieces with the property of Baire.*

($x \mapsto g \cdot x$ is a Borel automorphism if $A \in \mathcal{B}(X) \Rightarrow g \cdot A \in \mathcal{B}(X)$.)

However, there is a group action by Borel automorphisms on a Polish space such and two Borel sets B_0, B_1 such that $B_0 \sim_G B_1$ but not by sets with the property of Baire.

27.3 De Groot's Problem

Problem 27.10 (De Groot, 1958)**.** *Can Banach–Tarski duplication be done so that the pieces are moved continuously without overlapping? More precisely, say $A, B \subseteq \mathbb{R}^n$ are* continuously equidecomposable *(write $A \sim^c B$) if $A = \bigsqcup_{i=1}^k A_i, B = \bigsqcup_{i=1}^k B_i$ and there are continuous G_n-paths $\gamma^i : [0,1] \to G_n$ such that $g^i(0) = \mathrm{id}$ and $g^i(1)(A_i) = B_i$. Additionally, we require that for all $t \in [0,1]$, $\gamma^i(t)(A_i) \cap \gamma^j(t)(A_j) = \emptyset$ for $i \neq j$.*

Theorem 27.11 (Trevor Wilson, 2005)**.** *Let $n \geq 2$. If $A, B \subseteq \mathbb{R}^n$ are bounded, and $A \sim B$, then $A \sim^c B$.*